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Double-quantitative decision-theoretic approach to multigranulation approximate space ☆

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ABSTRACT

The decision-theoretic rough set, as a special case of probabilistic rough set, mainly utilizes conditional probability to express relative quantitative information, while the graded rough set is characterized by absolute quantitative information between the partitions and basic concept. Thus, the double-quantification integrating relative and absolute quantitative information has become a fundamental topic for model construction, especially for developing the decision-theoretic rough set. In this study, we propose a basic framework of double-quantitative decision-theoretic rough set based on Bayesian decision and graded rough set approach in multigranulation approximate space. Three pairs of double-quantitative multigranulation decision-theoretic rough set models are established, which consist of a dual of optimistic double-quantitative multigranulation decision-theoretic rough sets, pessimistic double-quantitative multigranulation decision-theoretic rough sets and mean double-quantitative multigranulation decision-theoretic rough sets. These models essentially indicate the relative and absolute information quantification. Furthermore, some essential properties of these models are addressed and the decision rules which incorporate the relative and absolute quantitative information are investigated. Finally, an illustrative case about medical diagnosis is conducted to interpret and evaluate the double-quantitative decision-theoretic approach.

1. Introduction

Rough set (RS) theory which was originated by Pawlak [24,26], is an extension of the classical set theory and could be regarded as a mathematical and soft computing tool to handle imprecision, vagueness and uncertainty in data analysis. It has become a well-established theory for uncertainty management in a wide variety of applications related to pattern recognition [37], information fusion [8], feature selection [11,12], uncertainty analysis [5,17], rule learning [15], data modeling [36], and knowledge discovery [57]. Given there are no fault tolerance mechanisms between equivalence classes and basic concept set, several proposals of generalized quantitative rough set models were developed to resolve this limitation by using a graded set inclusion. The probabilistic rough set (PRS) introduces the probability uncertainty measure into RS [25], which forms the basis of mainstream quantitative models [1,21,23,40,52,53,59]. PRS offers measurability, generality,

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and flexibility and exhibits a series of concrete models which consist of the decision-theoretic rough set (DTRS) [49], game-theoretic rough set [2], variable rough set [58], Bayesian rough set [39], and parameter rough set [10]. They aim at modeling data relationships expressed in terms of frequency distribution rather than in terms of a full inclusion relation. With the exception of PRS, the graded rough set (GRS) depends on two absolute measures becomes another basic type of quantitative model [50].

In the viewpoint of information quantification, the DTRS and GRS can respectively reflect relative and absolute quantitative information about the degree of overlap between equivalence classes and concept set [13,62]. The relative and absolute quantitative information are two distinctive objective sides that describe approximate space, and each has its own virtues and application environments, so that none can be neglected. Here, we illustrate three examples to highlight the significance of combining the relative quantification and absolute quantification, and the necessity of these two types of quantitative model is exhibited in different scenarios.

(1) There is a good project that needs to be invested due to the lack of funds. So decision makers prepare to attract 20 million dollars to support the project implementation, and there are two mutually exclusive investment companies Alpha and Beta as alternatives that means only one investment company will be selected. According to the financial report, Alpha has a 50 million dollars budget available for investment and plans to invest 15 million dollars in this project, while Beta has a 30 million dollars budget available for investment and prepares to invest 12 million dollars in this project. So, which one is more suitable as a partner? There is no doubt that Alpha is the preferable choice, although the relative proportion of budget is only 30%, which is lower than that 40% of Beta. Here, one focus mainly on the absolute quantitative information and believes that lower priority of the relative quantitative information.
(2) A company is ready to purchase a large quantity of products in the near future. A and B are the suppliers of this product, and the price difference between them is tiny. Therefore the chief executive officer prepares to conduct a selective examination of product quality to determine who will win the bid. The results of sample survey show that company A has 15 substandard products in the 300 samples and company B has 20 substandard products in 1000 samples. It is clearly that the number of unqualified products of A than B more ($15 < 20$). But we still believe that B is a better candidate due to the product qualified rate is $98\% > 95\%$. As this example suggests, the higher priority should be the relative quantitative information not the absolute quantitative information.
(3) Suppose A and B are research institutes with 70 and 30 proposed projects, respectively, but only 50 projects will be approved and funded in total. How does one make a final decision on which projects to implement? If only the relative quantitative information is considered, one may conclude that A and B will achieve 35 and 15 establishment projects, respectively. However, is this division fair and reasonable in reality? It may be feasible if the two institutes with almost the same scientific research ability. However, if the research level of A is much higher than B, then A should obtain more than 35 projects and B should receive less than 15 projects. In practice, the absolute quantitative information that the number of approved projects is a pivotal index. Therefore, an ideal evaluation must employ rational combinations of the two evaluation indexes.

The relative and absolute measures adopt different quantitative views for measurement, thus underlying quantitative applications. Usually, both hold heterogeneity and complementarity, and thus, each relies on its essential benefit to occupy its own dominant environment. In recent decades, a lot of research interests are attracted by the double-quantitative fusion of relative quantitative information and absolute quantitative information. Zhang developed a comparative study of relative quantitative rough set model and absolute quantitative rough set model [62], then he systematically researched the issues of double-quantitative fusion [63–66]. Based on these achievements, Li constructed two double-quantitative decision-theoretic rough set models [13], Fan studied this issue based on logical conjunction and logical disjunction operation [6] and Fang proposed another kind of double relative quantitative decision-theoretic rough set models, which essentially indicate the relative and absolute quantification [7].

The granular computing (GrC), another powerful tool in artificial intelligence and data processing, which is a term coined jointly by Zadeh [60,61] and Lin [18]. Bargiela [3,4] and Pedrycz [28–30] conducted a series of systematic studies on GrC and many constructive achievements were obtained. In the view of GrC, a general concept described by a set is always characterized via the so-called upper and lower approximations under a single granulation, namely, the concept is depicted by knowledge induced from a single binary relation on the universe of discourse [51]. In many practical circumstances, we need to describe concurrently a target concept through multi binary relations according to users' requirements and targets of problem solving. Based on this thought, Qian et al. first investigated multigranulation rough set (MGRS) theory to more widely apply rough set theory [31], and introduced the incomplete multigranulation rough set [32]. Since the MGRS was established, the theoretic framework have been largely enriched, and many generalized MGRS models and their applications have also been investigated [33,34]. Wu extended classical MGRS to a novel version based on a fuzzy binary relation [41]. She explored the topological structures and the essential properties of MGRS [38], and Yang revealed the hierarchical structures properties of the MGRS [48]. Furthermore, Wu and Leung proposed a formal approach to granular computing with multi-scale data measured at different levels of granulations [42], Lin applied this method to information fusion by combining with evidence theory [19]. Prior to this study, we have expanded the classical MGRS model to a generalized formal [44], and developed the MGRS approach in fuzzy tolerance approximation space [45] and ordered information system [46,55,56], respectively. These studies provide an abundant theoretical basis for studying the approach of double-quantitative decision-theoretic in multigranulation approximate space.

Since Qian studied the multigranulation decision-theoretic rough set by combining multigranulation idea and Bayesian decision theory [35]. A series of generalized multigranulation decision-theoretic rough sets and their applications are discussed [9,14,16,20,22,47]. However, to the best of our knowledge, there are a lot of researches on decision-theoretic rough set but few studies on the double-quantitative decision-theoretic approach in the context of multigranulation approximate space. In numerous circumstances, there are some issues that not only the relative quantitative information but also the absolute quantitative information should be considered. Meanwhile we need to describe concurrently a target concept through multi binary relations according to users' requirements and targets of problem solving. Therefore, the motivation of this investigation is to develop a new double-quantitative multigranulation rough decision approach by combining the graded rough set and decision-theoretic rough set in multigranulation approximate space. There are three pairs of double-quantitative multigranulation decision-theoretic rough set models be established which consist of two optimistic double-quantitative multigranulation decision-theoretic rough sets, two pessimistic double-quantitative multigranulation decision-theoretic rough sets and two mean double-quantitative multigranulation decision-theoretic rough sets.

The remainder of this paper is organized as follows. In Section 2, some basic concepts are briefly reviewed. In Section 3, we establish several novel double-quantitative multigranulation decision-theoretic rough set models, the properties of these models are addressed and the decision rules are investigated. An illustrated case is conducted to evaluate the proposed double-quantitative multigranulation decision-theoretic approach and some decision rules are exhibited in Section 4. Finally, the paper ends with conclusions shown in Section 5.

2. Preliminaries

In this section, we briefly introduce some necessary notions which consist of rough set, decision-theoretic rough set, graded rough set and multigranulation rough set. It should be noted that $\mathcal{P}(U)$ is the power set of U, the $\sim X$ and X^C are the complement of X, and $|X|$ means the cardinality of set X throughout this paper. More details can refer to the references that we cited.

2.1. Pawlak rough set

An information system is represented as a quadruple $I = (U, AT, V, f)$, where U is a finite non-empty set of objects, AT is a finite non-empty set of attributes, V is a set of attribute value and f is a mapping which from U to V, the $f_a(x)$ means the attribute value of x with respect to a. In rough set theory, an equivalence relation (indiscernibility relation) is the foundation of classification mechanism. We can define an indiscernibility relation $IND(A)$ with respect to A (where $A \subseteq AT$) as follows:

$$IND(A) = \{(x, y) \in U \times U : f_a(x) = f_a(y), a \in A\}. \tag{2.1}$$

According to the indiscernibility relation $IND(A)$, we can obtain the equivalence class containing x by following way:

$$[x]_A = \{y \in U : (x, y) \in IND(A)\}. \tag{2.2}$$

It implies that the quotient set of U is a partition of the universe. In the view of GrC, equivalence classes are the basic building blocks for the representation and approximation of concept. Each equivalence class may be viewed as a granule consisting of indistinguishable elements. For any basic concept $X \in \mathcal{P}(U)$, one can characterize X by a pair of lower and upper approximations with respect to A as follows:

$$\underline{A}(X) = \{x \in U : [x]_A \subseteq X\}, \tag{2.3}$$

$$\overline{A}(X) = \{x \in U : [x]_A \cap X \neq \emptyset\}. \tag{2.4}$$

Then, we can obtain the rough regions based on this definition. They are positive region $pos(X) = \underline{A}(X)$, negative region $neg(X) =\sim \overline{A}(X)$ and boundary region $bn(X) = \overline{A}(X) - \underline{A}(X)$ with respect to A, respectively.

2.2. Decision-theoretic rough set

In order to establish an fault tolerance mechanism between the equivalence classes and basic concept set, Pawlak and Skowron [27] suggested using a rough membership function to redefine the two approximations and the rough membership function $P(X|[x]_A)$ is defined as follows:

$$P(X|[x]_A) = \frac{|[x]_A \cap X|}{|[x]_A|}. \tag{2.5}$$

Table 1
The loss function.

	X (P)	X^C (N)
a_P	λ_{PP}	λ_{PN}
a_N	λ_{NP}	λ_{NN}
a_B	λ_{BP}	λ_{BN}

Analogously, Ziarko [58] defined an misclassification rate by following way:

$$c([x]_A, X) = 1 - \frac{|[x]_A \cap X|}{|[x]_A|}. \tag{2.6}$$

In fact, the $P(X|[x]_A)$ and $c([x]_A, X)$ refer to the relative overlap rate and relative errors with respect to knowledge A and concept set X, respectively. In the Bayesian decision produce, a finite set of states can be written as $\Omega = \{\omega_1, \omega_2, \cdots, \omega_s\}$, and a finite set of m possible actions can be denoted by $A = \{a_1, a_2, \cdots, a_r\}$. Let $P(\omega_j|x)$ be the conditional probability of an object x being in state ω_j given that the object is described by $\mathbf{x}$. Let $\lambda(a_i|\omega_j)$ denote the loss, or cost for taking action a_i when the state is ω_j, the expected loss function associated with taking action a_i is given by:

$$R(a_i|x) = \sum_{j=1}^{s} \lambda(a_i|\omega_j) P(\omega_j|\mathbf{x}). \tag{2.7}$$

With respect to the membership of an object in X, we have a set of two states and a set of three actions for each state. The set of states is given by $\Omega = \{X, X^C\}$ indicating that an element is in X or not in X, respectively. The set of actions with respect to a state is given by $A = \{a_P, a_B, a_N\}$, where P, B and N represent the three actions in deciding $x \in pos(X)$, deciding $x \in bn(X)$, and deciding $x \in neg(X)$, respectively. The loss function regarding the risk or cost of actions in different states is given in Table 1.

In the Table 1, λ_{PP}, λ_{NP} and λ_{BP} denote the losses incurred for taking actions a_P, a_N and a_B when an object belongs to X, and λ_{PN}, λ_{NN} and λ_{BN} denote the losses incurred for taking the same actions when the object does not belong to X, respectively. The expected loss $R(a_i|[x]_A)$ associated with taking the individual actions can be expressed in [5].

$$R(a_P|[x]_A) = \lambda_{PP} P(X|[x]_A) + \lambda_{PN} P(X^C|[x]_A),$$
$$R(a_N|[x]_A) = \lambda_{NP} P(X|[x]_A) + \lambda_{NN} P(X^C|[x]_A),$$
$$R(a_B|[x]_A) = \lambda_{BP} P(X|[x]_A) + \lambda_{BN} P(X^C|[x]_A).$$

When $\lambda_{PP} \le \lambda_{NP} < \lambda_{BP}$ and $\lambda_{BN} \le \lambda_{NN} < \lambda_{PN}$, the Bayesian decision procedure leads to the following minimum-risk decision rules:

(P) If $P(X|[x]_A) \ge \gamma$ and $P(X|[x]_A) \ge \alpha$, decide $pos(X)$;
(N) If $P(X|[x]_A) \le \beta$ and $P(X|[x]_A) \le \gamma$, decide $neg(X)$;
(B) If $\beta \le P(X|[x]_A) \le \alpha$, decide $bn(X)$.

Where the parameters α, β and γ are defined as:

$$\alpha = \frac{\lambda_{PN} - \lambda_{BN}}{(\lambda_{PN} - \lambda_{BN}) + (\lambda_{BP} - \lambda_{PP})}, \tag{2.8}$$

$$\beta = \frac{\lambda_{BN} - \lambda_{NN}}{(\lambda_{BN} - \lambda_{NN}) + (\lambda_{NP} - \lambda_{BP})}, \tag{2.9}$$

$$\gamma = \frac{\lambda_{PN} - \lambda_{NN}}{(\lambda_{PN} - \lambda_{NN}) + (\lambda_{NP} - \lambda_{PP})}. \tag{2.10}$$

If a loss function further satisfies the condition that $(\lambda_{PN} - \lambda_{BN})(\lambda_{NP} - \lambda_{BP}) \ge (\lambda_{BN} - \lambda_{NN})(\lambda_{BP} - \lambda_{PP})$, then we can get $\alpha \ge \gamma \ge \beta$. Moreover, we can get that $\alpha > \gamma > \beta$ if $\alpha > \beta$, thus, the DTRS has the decision rules:

(P) If $P(X|[x]_A) \ge \alpha$, decide $pos(X)$;
(N) If $P(X|[x]_A) \le \beta$, decide $neg(X)$;
(B) If $\beta < P(X|[x]_A) < \alpha$, decide $bn(X)$.

Using these decision rules, we get the probabilistic approximations, namely, the lower and upper approximations of DTRS model as follows:

$$\underline{A}_{(\alpha,\ \beta)}(X) = \{x \in U : P(X|[x]_A) \ge \alpha\}, \tag{2.11}$$
$$\overline{A}_{(\alpha,\ \beta)}(X) = \{x \in U : P(X|[x]_A) > \beta\}. \tag{2.12}$$

Here, $pos_{(\alpha,\ \beta)}(X) = \underline{A}_{(\alpha,\ \beta)}(X)$, $neg_{(\alpha,\ \beta)}(X) = \sim \overline{A}_{(\alpha,\ \beta)}(X)$, $bn_{(\alpha,\ \beta)}(X) = \overline{A}_{(\alpha,\ \beta)}(X) - \underline{A}_{(\alpha,\ \beta)}(X)$ are the positive region, negative region and boundary region, respectively.

2.3. Graded rough set

The absolute quantitative information between basic concept and knowledge granules is the main investigation topic of the GRS. Hence, Yao and Lin proposed the GRS model based on graded modal logics in [50]. For any concept set $X \in \mathcal{P}(U)$, suppose $k \in \mathbf{N}$ is a non-negative integer called "grade" and the lower and upper approximations are defined by following way:

$$\underline{A}_k(X) = \{x \in U : |[x]_A| - |[x]_A \cap X| \le k\}, \tag{2.13}$$

$$\overline{A}_k(X) = \{x \in U : |[x]_A \cap X| > k\}. \tag{2.14}$$

These two approximations are called grade k lower and upper approximations of X with respect to A. The lower approximation set is an set of elements that satisfy the cardinality of the intersection of the complementary set of X and objects not to exceed parameter k. On the other hand, the upper approximation set means an set of elements which satisfy the cardinality of the intersection of X and equivalence classes to exceed parameter k. The grade k positive region, negative region, lower boundary region and upper boundary region of X are $pos(X) = \overline{A}_k(X) \cap \underline{A}_k(X)$, $neg(X) = \sim (\overline{A}_k(X) \cup \underline{A}_k(X))$, $Lbn(X) = \underline{A}_k(X) - \overline{A}_k(X)$ and $Ubn(X) = \overline{A}_k(X) - \underline{A}_k(X)$, respectively. The boundary region is the symmetric difference of lower and upper approximation set.

$$bn(X) = \overline{A}_k(X) \triangle \underline{A}_k(X). \tag{2.15}$$

Where the "$\triangle$" is the symmetric difference of sets that means $bn(X) = Lbn_k(X) \cup Ubn_k(X)$. Here, it should be noted that the lower approximation included in the upper approximation does not hold in usually.

2.4. Multigranulation rough set

In order to solve the problem which induced by a family of indiscernibility relation instead of single equivalence relations, Qian et al. developed two different multigranulation rough set including the optimistic and pessimistic cases. Let I be an information system in which $A_1, A_2, \cdots, A_m \subseteq AT$, for any $X \in \mathcal{P}(U)$, the optimistic multigranulation lower and upper approximations are denoted by:

$$\underline{\sum_{i=1}^{m} A_i}^{O}(X) = \{x \in U : \bigvee_{i=1}^{m}([x]_{A_i} \subseteq X)\}, \tag{2.16}$$

$$\overline{\sum_{i=1}^{m} A_i}^{O}(X) = \sim \underline{\sum_{i=1}^{m} A_i}^{O}(\sim X). \tag{2.17}$$

Where the $[x]_{A_i}$ means the equivalence class of x in terms of attributes set A_i and $i = 1, 2, \cdots, m$. Furthermore, we can get that the optimistic multigranulation upper approximation $\overline{\sum_{i=1}^{m} A_i}^{O}(X) = \{x \in U : \bigwedge_{i=1}^{m}([x]_{A_i} \cap X \neq \emptyset)\}$. It can be considered as a set in which objects have non-empty intersection with the target in terms of each granular structure. On the other strategy, the definition of pessimistic multigranulation rough set can be given as follows:

$$\underline{\sum_{i=1}^{m} A_i}^{P}(X) = \{x \in U : \bigwedge_{i=1}^{m}([x]_{A_i} \subseteq X)\}, \tag{2.18}$$

$$\overline{\sum_{i=1}^{m} A_i}^{P}(X) = \sim \underline{\sum_{i=1}^{m} A_i}^{P}(\sim X). \tag{2.19}$$

Analogously, the pessimistic multigranulation upper approximation can be described as $\overline{\sum_{i=1}^{m} A_i}^{P}(X) = \{x \in U : \bigvee_{i=1}^{m}([x]_{A_i} \cap X \neq \emptyset)\}$. Different from the upper approximation of optimistic multigranulation rough set, the upper approximation of pessimistic multigranulation rough set is represented as a set in which objects have non-empty intersection with the target in terms of at least one granular structure.

3. Double-quantitative multigranulation decision-theoretic rough set

In [35], Qian et al. introduced there exist three cases of decision-theoretic rough sets in a multigranulation approximate space. They are optimistic, pessimistic and mean multigranulation decision-theoretic rough sets, respectively. On the other hand, Li [13] concluded that there are two novel scenarios will be generated by recombining the absolute and relative quantitative approximation operators. Therefore, there are six kinds of scenarios to establish double-quantitative decision-theoretic rough set models by recombining the absolute and relative quantitative approximation operators in a multigranulation approximate space. In this section, we will try to discuss these double-quantitative multigranulation decision-theoretic rough set models, which essentially incorporate the relative and absolute quantitative information. Before modeling, the feasibility analysis of model establishment is presented and internal relationship between quantitative variables is discussed.

3.1. Double-quantification

Due to the relative and absolute errors, two basic notions, are widely existed in numerous measurements. In quantitative rough set model, Zhang [64] concluded that the $c([x]_A, X)$ (misclassification rate) and $\underline{g}([x]_A, X)$ (external grade) are the relative and absolute errors with respect to knowledge A and concept set X. Meanwhile the $P(X|[x]_A)$ (conditional probability) and $\overline{g}([x]_A, X)$ (internal grade) can refer to the relative overlap rate and absolute overlap number, respectively. In fact, there are two and only two core measures that $|[x]_A|$ and $|[x]_A \cap X|$. Other quantitative variables can be formed based on these two core measures. Therefore, we can get the following formulas according to the description of quantitative variables.

$$\begin{cases} P(X|[x]_A) = \dfrac{|[x]_A \cap X|}{|[x]_A|}, \\ \underline{g}([x]_A, X) = |[x]_A| - |[x]_A \cap X|, \\ \overline{g}([x]_A, X) = |[x]_A \cap X|. \end{cases} \tag{3.1}$$

According to the formula (3.1), we can get the relationship between these measures as follows:

$$\overline{g}([x]_A, X) = \frac{P(X|[x]_A)}{1 - P(X|[x]_A)} \times \underline{g}([x]_A, X). \tag{3.2}$$

Proof. Based on the descriptions of formula (3.1), we can get that $\frac{\underline{g}([x]_A, X)}{\overline{g}([x]_A, X)} = \frac{|[x]_A| - |[x]_A \cap X|}{|[x]_A \cap X|} = \frac{1}{P(X|[x]_A)} - 1$. So, we can obtain that $\frac{\underline{g}([x]_A, X)}{\overline{g}([x]_A, X)} = \frac{1-P(X|[x]_A)}{P(X|[x]_A)}$, that is to say, $\overline{g}([x]_A, X) = \frac{P(X|[x]_A)}{1-P(X|[x]_A)} \times \underline{g}([x]_A, X)$. Here, it should be noted that all of the fractions are well-defined in this process. This completes the proof. □

In the previous proof, we can get another forms of these core measures that act as $|[x]_A| = \frac{\underline{g}([x]_A, X)}{1-P(X|[x]_A)}$ and $|[x]_A \cap X| = \frac{P(X|[x]_A)}{1-P(X|[x]_A)} \times \underline{g}([x]_A, X)$. They imply that the following formulas hold.

$$P(X|[x]_A) = \frac{\overline{g}([x]_A, X)}{|[x]_A|}, \tag{3.3}$$

$$P(X|[x]_A) = 1 - \frac{\underline{g}([x]_A, X)}{|[x]_A|}. \tag{3.4}$$

Proof. They can be proved directly by formula (3.1). □

The formulas (3.2)–(3.4) show that the conditional probability and grades (external grade and internal grade) are mutually dependent but nonlinear. The conditional probability and grades are related to the relative and absolute quantitative information, respectively. But, these indexes are not equivalent, and the relationship between them is often close, complementary and dialectical. Therefore, by incorporating the conditional probability and grade, the DTRS and GRS further contribute to the relative and absolute quantification, thus having relative and absolute fault-tolerance capabilities. The double-quantification method can provide a more thorough description of the approximate space and promote knowledge discovery based to double-quantitative information. Normally, there are two ways to establish double-quantitative rough set model, which consist of the logical operations [6] and the combination of upper and lower approximation operators [13,43,63]. In this talk, we will try to build the double-quantitative decision-theoretic rough set model in multigranulation approximate space by the combination of upper and lower approximation operators. Similarly to the classical multigranulation decision-theoretic rough set the discussion are divided into optimistic, pessimistic and mean scenarios, respectively.

3.2. Optimistic double-quantitative multigranulation decision-theoretic rough set

In existing optimistic multigranulation rough set approaches, the world "optimistic" is used to express the idea that in multiple independent granular structures, its lower approximation only needs at least one granular structure to satisfy with the inclusion condition between an equivalence class and the concept set. Qian [35] pointed out that the lower approximation collects those objects in which each object has at least one granular structure satisfying the probability constraint ($\geq \alpha$) between its equivalence class and the concept set, while the upper approximation collects those objects in which each object has all granular structures satisfying the probability constraint ($> \beta$) between the equivalence and the concept set. In the viewpoint of decision-making, the optimistic multigranulation decision-theoretic rough set is applicable to the scenarios with few requirements. It means that a small number of conditions should be satisfied to obtain decision rules in applications. Here, we will define the optimistic double-quantitative multigranulation decision-theoretic rough set by combining the rough approximation operators of DTRS and GRS in multigranulation approximate space.

Definition 3.1. (Dq-MDTRS$_{\mathrm{I}}^{O}$) Let $I = (U, AT, V, f)$ be an information system, given $A_1, A_2, \cdots, A_m \subseteq AT$ are granular structures, for any $X \in \mathcal{P}(U)$, $\beta \leq \alpha \in [0, 1]$, and $k \in \mathbf{N}$. Then, the lower and upper approximations of the first kind of optimistic double-quantitative multigranulation decision-theoretic rough set are denoted by $\underline{\sum_{i=1}^{m} A_i}_k^{O}(X)$ and $\overline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^{O}(X)$, respectively.

$$\underline{\sum_{i=1}^{m} A_i}_k^{O}(X) = \{x \in U : \bigvee_{i=1}^{m} (|[x]_{A_i}| - |[x]_{A_i} \cap X| \leq k)\}, \tag{3.5}$$

$$\overline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^{O}(X) = \{x \in U : \bigwedge_{i=1}^{m} (P(X|[x]_{A_i}) > \beta)\}. \tag{3.6}$$

The model $(U, \underline{\sum_{i=1}^{m} A_i}_k^{O}(X), \overline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^{O}(X))$ is the first kind of optimistic double-quantitative multigranulation decision-theoretic rough set and is marked as Dq-MDTRS$_{\mathrm{I}}^{O}$. Based on this pair of approximation operators, the positive region, negative region, upper boundary region and lower boundary can be achieved as follows:

(1) $pos_{\mathrm{I}}^{O}(X) = \underline{\sum_{i=1}^{m} A_i}_k^{O}(X) \cap \overline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^{O}(X)$;

(2) $neg_{\mathrm{I}}^{O}(X) = \sim (\underline{\sum_{i=1}^{m} A_i}_k^{O}(X) \cup \overline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^{O}(X))$;

(3) $Ubn_{\mathrm{I}}^{O}(X) = \overline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^{O}(X) - \underline{\sum_{i=1}^{m} A_i}_k^{O}(X)$;

(4) $Lbn_{\mathrm{I}}^{O}(X) = \underline{\sum_{i=1}^{m} A_i}_k^{O}(X) - \overline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^{O}(X)$.

Based on Definition 3.1, we can obtain some propositions of Dq-MDTRS$_{\mathrm{I}}^{O}$ which are represented as follows:

Proposition 3.1. *Given $A_1, A_2, \cdots, A_m \subseteq AT$ are granular structures, for any $X \in \mathcal{P}(U)$, $\beta \leq \alpha \in [0, 1]$, $k \in \mathbf{N}$. Then, the following properties hold.*

(1) $\underline{\sum_{i=1}^{m} A_i}^{O}(X) \supseteq \underline{A_i}_k(X)$;

(2) $\overline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^{O}(X) \subseteq \overline{A_i}_{(\alpha,\ \beta)}(X)$;

(3) $\underline{\sum_{i=1}^{m} A_i}^{O}(X) = \bigcup_{i=1}^{m} \underline{A_i}_k(X)$;

(4) $\overline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^{O}(X) = \bigcap_{i=1}^{m} \overline{A_i}_{(\alpha,\ \beta)}(X)$;

(5) $\underline{\sum_{i=1}^{m} A_i}_k^{O}(X) \subseteq \underline{\sum_{i=1}^{m} A_i}_k^{O}(Y)$, *if* $X \subseteq Y \in \mathcal{P}(U)$;

(6) $\overline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^{O}(X) \subseteq \overline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^{O}(Y)$, *if* $X \subseteq Y \in \mathcal{P}(U)$;

(7) $\underline{\sum_{i=1}^{m} A_i}_{k_1}^{O}(X) \subseteq \underline{\sum_{i=1}^{m} A_i}_{k_2}^{O}(X)$, *if* $k_1 \leq k_2 \in \mathbf{N}$;

(8) $\overline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta_1)}^{O}(X) \supseteq \overline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta_2)}^{O}(X)$, *if* $\beta_1 \leq \beta_2 \in [0, 1]$.

Proof. According to the formulas (2.11) and (2.13), we have $\underline{A_i}_k(X) = \{x \in U : |[x]_{A_i}| - |[x]_{A_i} \cap X| \leq k\}$ and $\overline{A_i}_{(\alpha,\ \beta)}(X) = \{x \in U : P(X|[x]_{A_i}) > \beta\}$. Then, these propositions can be proved directly from Definition 3.1. □

The Dq-MDTRS$_{\mathrm{I}}^{O}$ is a generalization of optimistic multigranulation rough set model. According to Definition 3.1, we can get $\underline{\sum_{i=1}^{m} A_i}_k^{O}(X) = \underline{\sum_{i=1}^{m} A_i}^{O}(X)$ if $k = 0$ and $\overline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^{O}(X) = \overline{\sum_{i=1}^{m} A_i}^{O}(X)$ if $\beta = 0$. That means the model with its thresholds exhibit favorable directional expansion properties.

The relative information similarly complements the absolute description and can be used to improve the GRS model. The sharp contrast between the relative and grade environments is typical of double quantification applications. For example, the relative quantification varies over a small range while the grade changes significantly, then the double quantification can play a significant role. Based on the descriptions of the regions and Proposition 3.1, the decision rules of Dq-MDTRS$_{\mathrm{I}}^{O}$ can be obtained as follows:

(P_{I}^{O}) If $\exists\ A_i \in \mathscr{A}$ such that $|[x]_{A_i}| - |[x]_{A_i} \cap X| \leq k$ and $\forall\ A_j \in \mathscr{A}$ such that $P([x]_{A_i}|X) > \beta$, decide $pos_{\mathrm{I}}^{O}(X)$.

(N_{I}^{O}) If $\forall\ A_i \in \mathscr{A}$ such that $|[x]_{A_i}| - |[x]_{A_i} \cap X| > k$ and $\exists\ A_j \in \mathscr{A}$ such that $P([x]_{A_i}|X) \leq \beta$, decide $neg_{\mathrm{I}}^{O}(X)$.

(UB_{I}^{O}) If $\forall\ A_i \in \mathscr{A}$ such that $|[x]_{A_i}| - |[x]_{A_i} \cap X| > k$ and $\forall\ A_j \in \mathscr{A}$ such that $P([x]_{A_i}|X) > \beta$, decide $Ubn_{\mathrm{I}}^{O}(X)$.

(LB_{I}^{O}) If $\exists\ A_i \in \mathscr{A}$ such that $|[x]_{A_i}| - |[x]_{A_i} \cap X| \leq k$ and $\exists\ A_j \in \mathscr{A}$ such that $P([x]_{A_i}|X) \leq \beta$, decide $Leg_{\mathrm{I}}^{O}(X)$.

Where the $i, j = 1, 2, \cdots, m$, and the i and j are irrelevant. It should be noted that $\mathscr{A} = \{A_1, A_2, \cdots, A_m\}$ is the set of predefined granular structures throughout this paper. Corresponding to the first kind of optimistic double-quantitative multigranulation decision-theoretic rough set model, we can define the anther model by following way.

Definition 3.2. (Dq-MDTRS$_{\mathrm{II}}^{O}$) Let $I = (U, AT, V, f)$ be an information system, given $A_1, A_2, \cdots, A_m \subseteq AT$ are granular structures, for any $X \in \mathcal{P}(U)$, $\beta \leq \alpha \in [0, 1]$, and $k \in \mathbf{N}$. Then, the second kind of optimistic double-quantitative multigranulation decision-theoretic rough set approximations are denoted as follows:

$$\underline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^{O}(X) = \{x \in U : \bigvee_{i=1}^{m} (P(X|[x]_{A_i}) \geq \alpha)\}, \tag{3.7}$$

$$\overline{\sum_{i=1}^{m} A_i}_{k}^{O}(X) = \{x \in U : \bigwedge_{i=1}^{m} (|[x]_{A_i} \cap X| > k)\}. \tag{3.8}$$

Corresponding to the Dq-MDTRS$_{\mathrm{I}}^{O}$, it is the second kind of optimistic double-quantitative multigranulation decision-theoretic rough set and is marked as Dq-MDTRS$_{\mathrm{II}}^{O}$. According to the definitions of lower and upper approximations, we can define the positive region, negative region, upper boundary region and lower boundary by following way:

(1) $pos_{\mathrm{II}}^{O}(X) = \underline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^{O}(X) \cap \overline{\sum_{i=1}^{m} A_i}_{k}^{O}(X)$;

(2) $neg_{\mathrm{II}}^{O}(X) = \sim (\underline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^{O}(X) \cup \overline{\sum_{i=1}^{m} A_i}_{k}^{O}(X))$;

(3) $Ubn_{\mathrm{II}}^{O}(X) = \overline{\sum_{i=1}^{m} A_i}_{k}^{O}(X) - \underline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^{O}(X)$;

(4) $Lbn_{\mathrm{II}}^{O}(X) = \underline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^{O}(X) - \overline{\sum_{i=1}^{m} A_i}_{k}^{O}(X)$.

Analogously, we can achieve the following propositions for second type of double-quantitative optimistic multigranulation decision-theoretic rough set.

Proposition 3.2. *Given $A_1, A_2, \cdots, A_m \subseteq AT$ are granular structures, for any $X \in \mathcal{P}(U)$, $\beta \leq \alpha \in [0, 1]$, $k \in \mathbf{N}$. Then, the following properties hold.*

(1) $\underline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^{O}(X) \supseteq \underline{A_i}_{(\alpha,\ \beta)}(X)$;

(2) $\overline{\sum_{i=1}^{m} A_i}_{k}^{O}(X) \subseteq \overline{A_i}_{k}(X)$;

(3) $\underline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^{O}(X) = \bigcup_{i=1}^{m} \underline{A_i}_{(\alpha,\ \beta)}(X)$;

(4) $\overline{\sum_{i=1}^{m} A_i}_{k}^{O}(X) = \bigcap_{i=1}^{m} \overline{A_i}_{k}(X)$;

(5) $\underline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^{O}(X) \subseteq \underline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^{O}(Y)$, if $X \subseteq Y \in \mathcal{P}(U)$;

(6) $\overline{\sum_{i=1}^{m} A_i}_{k}^{O}(X) \subseteq \overline{\sum_{i=1}^{m} A_i}_{k}^{O}(Y)$, if $X \subseteq Y \in \mathcal{P}(U)$;

(7) $\underline{\sum_{i=1}^{m} A_i}_{(\alpha_1,\ \beta)}^{O}(X) \supseteq \underline{\sum_{i=1}^{m} A_i}_{(\alpha_2,\ \beta)}^{O}(Y)$, if $\alpha_1 \le \alpha_2 \in [0, 1]$;

(8) $\overline{\sum_{i=1}^{m} A_i}_{k_1}^{O}(X) \supseteq \overline{\sum_{i=1}^{m} A_i}_{k_2}^{O}(X)$, if $k_1 \le k_2 \in \mathbf{N}$.

Proof. It can be proved easily based on the proof of Proposition 3.1 and Definition 3.2. □

The Dq-MDTRS$_{\mathrm{II}}^{O}$ is also a generalization of optimistic multigranulation rough set model. According to the Definition 3.2, we can get $\underline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^{O}(X) = \underline{\sum_{i=1}^{m} A_i}^{O}(X)$ if $\alpha = 1$ and $\overline{\sum_{i=1}^{m} A_i}_{k}^{O}(X) = \overline{\sum_{i=1}^{m} A_i}^{O}(X)$ if $k = 0$.

Based on the previous discussions, we can obtain the decision rules for second kind of optimistic double-quantitative multigranulation decision-theoretic rough set and the decision rules of the Dq-MDTRS$_{\mathrm{II}}^{O}$ are listed as follows:

(P_{II}^{O}) If $\exists\ A_i \in \mathscr{A}$ such that $P([x]_{A_i}|X) \ge \alpha$ and $\forall\ A_j \in \mathscr{A}$ such that $|[x]_{A_j} \cap X| > k$, decide $pos_{\mathrm{II}}^{O}(X)$.

(N_{II}^{O}) If $\forall\ A_i \in \mathscr{A}$ such that $P([x]_{A_i}|X) < \alpha$ and $\exists\ A_j \in \mathscr{A}$ such that $|[x]_{A_j} \cap X| \le k$, decide $neg_{\mathrm{II}}^{O}(X)$.

(UB_{II}^{O}) If $\forall\ A_i \in \mathscr{A}$ such that $P([x]_{A_i}|X) < \alpha$ and $\forall\ A_j \in \mathscr{A}$ such that $|[x]_{A_j} \cap X| > k$, decide $Ubn_{\mathrm{II}}^{O}(X)$.

(LB_{II}^{O}) If $\exists\ A_i \in \mathscr{A}$ such that $P([x]_{A_i}|X) \ge \alpha$ and $\exists\ A_j \in \mathscr{A}$ such that $|[x]_{A_j} \cap X| \ge k$, decide $Lbn_{\mathrm{II}}^{O}(X)$.

According to Definitions 3.1 and 3.2, we can get that the optimistic double-quantitative multigranulation decision-theoretic rough set model are established based on approximation operators of GRS and DTRS in multigranulation approximation space. The Dq-MDTRS$_{\mathrm{I}}^{O}$ and Dq-MDTRS$_{\mathrm{II}}^{O}$ are generalizations of optimistic multigranulation rough set models. The Dq-MDTRS$_{\mathrm{I}}^{O}$ is defined with respect to the parameters k and β and the Dq-MDTRS$_{\mathrm{II}}^{O}$ is defined based on the thresholds α and k, respectively. Both of them can degenerate into classical optimistic multigranulation rough set if the parameters are special enough. These decision rules can be utilized to different fields based on different applied requirements.

3.3. Pessimistic double-quantitative multigranulation decision-theoretic rough set

In this subsection, we will establish two pessimistic double-quantitative multigranulation decision-theoretic rough set models and discuss the decision rules of these models. Corresponding to the optimistic multigranulation rough set approaches, the world "pessimistic" is utilized to characterize the idea in terms of multiple independent granular structures, the positive region should be satisfied with the inclusion condition between equivalence class and concept set with respect to all granular structures. In applications, this method is suitable for the situation with strict conditions, which means all requirements should be fulfilled to achieve decision rules. Similar to the double-quantitative optimistic multigranulation decision-theoretic rough set, the investigation is divided into two cases and some relevant decision rules are studied, respectively.

Definition 3.3. (Dq-MDTRS$_{\mathrm{I}}^{P}$) Let $I = (U, AT, V, f)$ be an information system, given $A_1, A_2, \cdots, A_m \subseteq AT$ are granular structures, for any $X \in \mathcal{P}(U)$, $\beta \le \alpha \in [0, 1]$, and $k \in \mathbf{N}$. Then, the lower and upper approximations of first kind of pessimistic double-quantitative multigranulation decision-theoretic rough set are denoted as follows:

$$\underline{\sum_{i=1}^{m} A_i}_{k}^{P}(X) = \{x \in U : \bigwedge_{i=1}^{m}(|[x]_{A_i}| - |[x]_{A_i} \cap X| \le k)\}, \tag{3.9}$$

$$\overline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^{P}(X) = \{x \in U : \bigvee_{i=1}^{m}(P(X|[x]_{A_i}) > \beta)\}. \tag{3.10}$$

This model is the first kind of pessimistic double-quantitative multigranulation decision-theoretic rough set and is marked as Dq-MDTRS$_{\mathrm{I}}^{P}$. Based on this pair of approximation operators, the positive region, negative region, upper boundary region and lower boundary can be achieved as follows:

(1) $pos_{\mathrm{I}}^{P}(X) = \underline{\sum_{i=1}^{m} A_i}_{k}^{P}(X) \cap \overline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^{P}(X)$;

(2) $neg_{\mathrm{I}}^P(X) = \sim(\underline{\sum_{i=1}^m A_i}_k^P(X) \cup \overline{\sum_{i=1}^m A_i}_{(\alpha,\ \beta)}^P(X))$;

(3) $Ubn_{\mathrm{I}}^P(X) = \overline{\sum_{i=1}^m A_i}_{(\alpha,\ \beta)}^P(X) - \underline{\sum_{i=1}^m A_i}_k^P(X)$;

(4) $Lbn_{\mathrm{I}}^P(X) = \underline{\sum_{i=1}^m A_i}_k^P(X) - \overline{\sum_{i=1}^m A_i}_{(\alpha,\ \beta)}^P(X)$.

According to Definition 3.3 and the description of these rough regions, we can achieve some propositions of the first kind of pessimistic double-quantitative multigranulation decision-theoretic rough set.

Proposition 3.3. *Given $A_1, A_2, \cdots, A_m \subseteq AT$ are granular structures, for any $X \in \mathcal{P}(U)$, $\beta \le \alpha \in [0, 1]$, $k \in \mathbf{N}$. Then, the following properties hold.*

(1) $\underline{\sum_{i=1}^m A_i}^P(X) \subseteq \underline{A_i}_k(X)$;

(2) $\overline{\sum_{i=1}^m A_i}_{(\alpha,\ \beta)}^P(X) \supseteq \overline{A_i}_{(\alpha,\ \beta)}(X)$;

(3) $\underline{\sum_{i=1}^m A_i}^P(X) = \bigcap_{i=1}^m \underline{A_i}_k(X)$;

(4) $\overline{\sum_{i=1}^m A_i}_{(\alpha,\ \beta)}^P(X) = \bigcup_{i=1}^m \overline{A_i}_{(\alpha,\ \beta)}(X)$;

(5) $\underline{\sum_{i=1}^m A_i}_k^P(X) \subseteq \underline{\sum_{i=1}^m A_i}_k^P(Y)$, *if $X \subseteq Y \in \mathcal{P}(U)$;*

(6) $\overline{\sum_{i=1}^m A_i}_{(\alpha,\ \beta)}^P(X) \subseteq \overline{\sum_{i=1}^m A_i}_{(\alpha,\ \beta)}^P(Y)$, *if $X \subseteq Y \in \mathcal{P}(U)$;*

(7) $\underline{\sum_{i=1}^m A_i}_{k_1}^P(X) \subseteq \underline{\sum_{i=1}^m A_i}_{k_2}^P(X)$, *if $k_1 \le k_2 \in \mathbf{N}$;*

(8) $\overline{\sum_{i=1}^m A_i}_{(\alpha,\ \beta_1)}^P(X) \supseteq \overline{\sum_{i=1}^m A_i}_{(\alpha,\ \beta_2)}^P(X)$, *if $\beta_1 \le \beta_2 \in [0, 1]$.*

Proof. These propositions can be proved directly based on Definition 3.3. □

The Dq-MDTRS$_{\mathrm{I}}^P$ is a generalization of pessimistic multigranulation rough set model. According to Definition 3.3, we can get that $\underline{\sum_{i=1}^m A_i}_k^P(X) = \underline{\sum_{i=1}^m A_i}^P(X)$ if $k = 0$ and $\overline{\sum_{i=1}^m A_i}_{(\alpha,\ \beta)}^P(X) = \overline{\sum_{i=1}^m A_i}^P(X)$ if $\beta = 0$.

In the Dq-MDTRS$_{\mathrm{I}}^P$ model, we can get that it is a natural expansion of classical pessimistic multigranulation rough set. The lower approximation and upper approximation of Dq-MDTRS$_{\mathrm{I}}^P$ model are with respect to parameter k and β, respectively. According to the representation of rough regions and Proposition 3.3, we can achieve the decision rules of Dq-MDTRS$_{\mathrm{I}}^P$ as follows:

(P_{I}^P) If $\forall\ A_i \in \mathscr{A}$ such that $|[x]_{A_i}| - |[x]_{A_i} \cap X| \le k$ and $\exists\ A_j \in \mathscr{A}$ such that $P([x]_{A_i}|X) > \beta$, decide $pos_{\mathrm{I}}^P(X)$.
(N_{I}^P) If $\exists\ A_i \in \mathscr{A}$ such that $|[x]_{A_i}| - |[x]_{A_i} \cap X| > k$ and $\forall\ A_j \in \mathscr{A}$ such that $P([x]_{A_i}|X) \le \beta$, decide $neg_{\mathrm{I}}^P(X)$.
(UB_{I}^P) If $\exists\ A_i \in \mathscr{A}$ such that $|[x]_{A_i}| - |[x]_{A_i} \cap X| > k$ and $\exists\ A_j \in \mathscr{A}$ such that $P([x]_{A_i}|X) > \beta$, decide $Ubn_{\mathrm{I}}^P(X)$.
(LB_{I}^P) If $\exists\ A_i \in \mathscr{A}$ such that $|[x]_{A_i}| - |[x]_{A_i} \cap X| \le k$ and $\forall\ A_j \in \mathscr{A}$ such that $P([x]_{A_i}|X) \le \beta$, decide $neg_{\mathrm{I}}^P(X)$.

Similar to this kind of pessimistic double-quantitative multigranulation decision-theoretic rough set model, we can establish the second type of pessimistic double-quantitative multigranulation decision-theoretic rough set model. It can be denoted by following way.

Definition 3.4. (Dq-MDTRS$_{\mathrm{II}}^P$) Let $I = (U, AT, V, f)$ be an information system, given $A_1, A_2, \cdots, A_m \subseteq AT$ are granular structures, for any $X \in \mathcal{P}(U)$, $\beta \le \alpha \in [0, 1]$, and $k \in \mathbf{N}$. Then, the second kind of pessimistic double-quantitative multigranulation decision-theoretic rough set approximations are defined by following way.

$$\underline{\sum_{i=1}^m A_i}_{(\alpha,\ \beta)}^P(X) = \{x \in U : \bigwedge_{i=1}^m (P(X|[x]_{A_i}) \ge \alpha)\}, \tag{3.11}$$

$$\overline{\sum_{i=1}^m A_i}_k^P(X) = \{x \in U : \bigvee_{i=1}^m (|[x]_{A_i} \cap X| > k)\}. \tag{3.12}$$

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We abbreviate this model as Dq-MDTRS$_{\text{II}}^{P}$. Analogously, the positive region, negative region, upper boundary region and lower boundary of Dq-MDTRS$_{\text{II}}^{P}$ can be obtained as follows:

(1) $pos_{\text{II}}^{P}(X) = \underline{\sum_{i=1}^{m} A_i}^{P}_{(\alpha,\ \beta)}(X) \cap \overline{\sum_{i=1}^{m} A_{i_k}}^{P}(X)$;

(2) $neg_{\text{II}}^{P}(X) = \sim (\underline{\sum_{i=1}^{m} A_i}^{P}_{(\alpha,\ \beta)}(X) \cup \overline{\sum_{i=1}^{m} A_{i_k}}^{P}(X))$;

(3) $Ubn_{\text{II}}^{P}(X) = \overline{\sum_{i=1}^{m} A_{i_k}}^{P}(X) - \underline{\sum_{i=1}^{m} A_i}^{P}_{(\alpha,\ \beta)}(X)$;

(4) $Lbn_{\text{II}}^{P}(X) = \underline{\sum_{i=1}^{m} A_i}^{P}_{(\alpha,\ \beta)}(X) - \overline{\sum_{i=1}^{m} A_{i_k}}^{P}(X)$.

From the definition of second type of pessimistic double-quantitative multigranulation decision-theoretic rough set, we can get the following propositions.

Proposition 3.4. *Given $A_1, A_2, \cdots, A_m \subseteq AT$ are granular structures, for any $X \in \mathcal{P}(U)$, $\beta \le \alpha \in [0, 1]$, $k \in \mathbf{N}$. Then, the following properties hold.*

(1) $\underline{\sum_{i=1}^{m} A_i}^{P}_{(\alpha,\ \beta)}(X) \subseteq \underline{A_i}_{(\alpha,\ \beta)}(X)$;

(2) $\overline{\sum_{i=1}^{m} A_{i_k}}^{P}(X) \supseteq \overline{A_i}_{k}(X)$;

(3) $\underline{\sum_{i=1}^{m} A_i}^{P}_{(\alpha,\ \beta)}(X) = \bigcap_{i=1}^{m} \underline{A_i}_{(\alpha,\ \beta)}(X)$;

(4) $\overline{\sum_{i=1}^{m} A_{i_k}}^{P}(X) = \bigcup_{i=1}^{m} \overline{A_i}_{k}(X)$;

(5) $\underline{\sum_{i=1}^{m} A_i}^{P}_{(\alpha,\ \beta)}(X) \subseteq \underline{\sum_{i=1}^{m} A_i}^{P}_{(\alpha,\ \beta)}(Y)$, *if* $X \subseteq Y \in \mathcal{P}(U)$;

(6) $\overline{\sum_{i=1}^{m} A_{i_k}}^{P}(X) \subseteq \overline{\sum_{i=1}^{m} A_{i_k}}^{P}(Y)$, *if* $X \subseteq Y \in \mathcal{P}(U)$;

(7) $\underline{\sum_{i=1}^{m} A_i}^{P}_{(\alpha_1,\ \beta)}(X) \supseteq \underline{\sum_{i=1}^{m} A_i}^{P}_{(\alpha_2,\ \beta)}(Y)$, *if* $\alpha_1 \le \alpha_2 \in [0, 1]$;

(8) $\overline{\sum_{i=1}^{m} A_{i_{k_1}}}^{P}(X) \supseteq \overline{\sum_{i=1}^{m} A_{i_{k_2}}}^{P}(X)$, *if* $k_1 \le k_2 \in \mathbf{N}$.

Proof. These propositions can be proved directly based on the introduction of Proposition 3.1 and Definition 3.4. □

Based on the Proposition 3.2 and Proposition 3.4, we can obtain that the monotonicity of Dq-MDTRS$_{\text{II}}^{O}$ accordance with monotonicity of Dq-MDTRS$_{\text{II}}^{P}$. That means the lower and upper approximations have the increase/decrease monotonicity with respect to the thresholds α and k, respectively.

The Dq-MDTRS$_{\text{II}}^{P}$ is a generalization of pessimistic multigranulation rough set model. According to Definition 3.4, we can achieve that $\underline{\sum_{i=1}^{m} A_i}^{P}_{(\alpha,\ \beta)}(X) = \underline{\sum_{i=1}^{m} A_i}^{P}(X)$ if $\alpha = 1$ and $\overline{\sum_{i=1}^{m} A_{i_k}}^{P}(X) = \overline{\sum_{i=1}^{m} A_i}^{P}(X)$ if $k = 0$. Based on the description of rough regions and Proposition 3.4, the decision rules of the Dq-MDTRS$_{\text{II}}^{P}$ can be achieved as follows:

(P_{II}^{P}) If $\forall\ A_i \in \mathscr{A}$ such that $P([x]_{A_i}|X) \ge \alpha$ and $\exists\ A_j \in \mathscr{A}$ such that $|[x]_{A_j} \cap X| > k$, decide $pos_{\text{II}}^{P}(X)$.

(N_{II}^{P}) If $\exists\ A_i \in \mathscr{A}$ such that $P([x]_{A_i}|X) < \alpha$ and $\forall\ A_j \in \mathscr{A}$ such that $|[x]_{A_j} \cap X| \le k$, decide $neg_{\text{II}}^{P}(X)$.

(UB_{II}^{P}) If $\exists\ A_i \in \mathscr{A}$ such that $P([x]_{A_i}|X) < \alpha$ and $\exists\ A_j \in \mathscr{A}$ such that $|[x]_{A_j} \cap X| > k$, decide $Ubn_{\text{II}}^{P}(X)$.

(LB_{II}^{P}) If $\forall\ A_i \in \mathscr{A}$ such that $P([x]_{A_i}|X) \ge \alpha$ and $\exists\ A_j \in \mathscr{A}$ such that $|[x]_{A_j} \cap X| \ge k$, decide $Lbn_{\text{II}}^{P}(X)$.

According to the decision rules of Dq-MDTRS$_{\text{II}}^{O}$ and Dq-MDTRS$_{\text{II}}^{P}$, we can see that these decision rules are symmetrical. That is because the symmetry of classical optimistic multigranulation rough set and pessimistic multigranulation rough set. These propositions indicate that some essential properties of the novel rough set model still hold. In the following study, we will define a new type of double-quantitative multigranulation decision-theoretic rough set model between optimistic double-quantitative multigranulation decision-theoretic rough set and pessimistic double-quantitative multigranulation decision-theoretic rough set.

3.4. Mean double-quantitative multigranulation decision-theoretic rough set

Analogously, this talk will be divided into two cases by combining the lower and upper approximations operators of GRS and DTRS in multigranulation approximation space. In classical multigranulation rough set approach, to describe the optimistic and pessimistic lower approximations. The word "optimistic" means that we need only at least one granular structure to satisfy with the inclusion condition between equivalence class and concept set, the word "pessimistic" is utilized to express the idea that with respect to all granular structures should be satisfied with the inclusion condition between equivalence class and concept set. That can be considered as maximum and minimum in the viewpoint of numerical calculation. However, these approaches seem too weak or strict in some practical applications. Consequently, we try to establish a mean double-quantitative multigranulation decision-theoretic rough set based on an average value.

Definition 3.5. (Dq-MDTRS$_{\mathrm{I}}^{M}$) Let $I=(U, AT, V, f)$ be an information system, given $A_1, A_2, \cdots, A_m \subseteq AT$ are granular structures, for any $X \in \mathcal{P}(U)$, $\beta \le \alpha \in [0,1]$, and $k \in \mathbf{N}$. Then, the lower and upper approximations of the first kind of mean double-quantitative multigranulation decision-theoretic rough set are defined as follows:

$$\underline{\sum_{i=1}^{m} A_i}_{k}^{M}(X)=\{x \in U : \frac{1}{m}\sum_{i=1}^{m}(|[x]_{A_i}|-|[x]_{A_i}\cap X|)\le k\}, \tag{3.13}$$

$$\overline{\sum_{i=1}^{m} A_i}^{M}_{(\alpha,\ \beta)}(X)=\{x \in U : \frac{1}{m}\sum_{i=1}^{m}(P(X|[x]_{A_i}))>\beta\}. \tag{3.14}$$

We call this model is the first kind of mean double-quantitative multigranulation decision-theoretic rough set model and abbreviate it as Dq-MDTRS$_{\mathrm{I}}^{M}$. Based on this pair of lower and upper approximation operators, the positive region, negative region, upper boundary region and lower boundary can be also obtained by following way.

(1) $pos_{\mathrm{I}}^{M}(X)=\underline{\sum_{i=1}^{m} A_i}_{k}^{M}(X)\cap\overline{\sum_{i=1}^{m} A_i}^{M}_{(\alpha,\ \beta)}(X)$;

(2) $neg_{\mathrm{I}}^{M}(X)=\sim(\underline{\sum_{i=1}^{m} A_i}_{k}^{M}(X)\cup\overline{\sum_{i=1}^{m} A_i}^{M}_{(\alpha,\ \beta)}(X))$;

(3) $Ubn_{\mathrm{I}}^{M}(X)=\overline{\sum_{i=1}^{m} A_i}^{M}_{(\alpha,\ \beta)}(X)-\underline{\sum_{i=1}^{m} A_i}_{k}^{M}(X)$;

(4) $Lbn_{\mathrm{I}}^{M}(X)=\underline{\sum_{i=1}^{m} A_i}_{k}^{M}(X)-\overline{\sum_{i=1}^{m} A_i}^{M}_{(\alpha,\ \beta)}(X)$.

According to Definition 3.5, we know that there is a difference between this model and classical multigranulation rough set model. Both the lower and upper approximations depend on a parameter that be induced by an average value of multi granular structures. Some essential mathematical properties of this model may be changed. Thus, we conduct an investigation on Dq-MDTRS$_{\mathrm{I}}^{M}$ and the following propositions are obtained.

Proposition 3.5. *Given $A_1, A_2, \cdots, A_m \subseteq AT$ are granular structures, for any $X \in \mathcal{P}(U)$, $\beta \le \alpha \in [0,1]$, $k \in \mathbf{N}$. Then, the following properties hold.*

(1) $\underline{\sum_{i=1}^{m} A_i}_{k}^{P}(X)\subseteq\underline{\sum_{i=1}^{m} A_i}_{k}^{M}(X)\subseteq\underline{\sum_{i=1}^{m} A_i}_{k}^{O}(X)$;

(2) $\overline{\sum_{i=1}^{m} A_i}^{P}_{(\alpha,\ \beta)}(X)\supseteq\overline{\sum_{i=1}^{m} A_i}^{M}_{(\alpha,\ \beta)}(X)\supseteq\overline{\sum_{i=1}^{m} A_i}^{O}_{(\alpha,\ \beta)}(X)$;

(3) $\underline{\sum_{i=1}^{m} A_i}_{k}^{M}(X)\subseteq\underline{\sum_{i=1}^{m} A_i}_{k}^{M}(Y)$, *if* $X \subseteq Y \in \mathcal{P}(U)$;

(4) $\overline{\sum_{i=1}^{m} A_i}^{M}_{(\alpha,\ \beta)}(X)\subseteq\overline{\sum_{i=1}^{m} A_i}^{M}_{(\alpha,\ \beta)}(Y)$, *if* $X \subseteq Y \in \mathcal{P}(U)$;

(5) $\underline{\sum_{i=1}^{m} A_i}_{k_1}^{M}(X)\subseteq\underline{\sum_{i=1}^{m} A_i}_{k_2}^{M}(X)$, *if* $k_1 \le k_2 \in \mathbf{N}$;

(6) $\overline{\sum_{i=1}^{m} A_i}^{M}_{(\alpha,\ \beta_1)}(X)\supseteq\overline{\sum_{i=1}^{m} A_i}^{M}_{(\alpha,\ \beta_2)}(X)$, *if* $\beta_1 \le \beta_2 \in [0,1]$.

Proof. (1) For any $x \in \underline{\sum_{i=1}^{m} A_i}_{k}^{P}(X)$, we know that for all $A_i \in \mathscr{A}$ have $|[x]_{A_i}|-|[x]_{A_i}\cap X|\le k$ where $i=1,2,\cdots,m$. So, we obtain that $\sum_{i=1}^{m}(|[x]_{A_i}|-|[x]_{A_i}\cap X|)\le m\cdot k$ means $\frac{1}{m}\sum_{i=1}^{m}(|[x]_{A_i}|-|[x]_{A_i}\cap X|)\le k$, that is $x \in \underline{\sum_{i=1}^{m} A_i}_{k}^{M}(X)$, namely, $\underline{\sum_{i=1}^{m} A_i}_{k}^{P}(X)\subseteq\underline{\sum_{i=1}^{m} A_i}_{k}^{M}(X)$. On the other hand, for any $x \in \underline{\sum_{i=1}^{m} A_i}_{k}^{M}(X)$, we can get that $\frac{1}{m}\sum_{i=1}^{m}(|[x]_{A_i}|-|[x]_{A_i}\cap X|)\le k$ based on Definition 3.5. It indicates that there is at least one granular structure A_i such that $|[x]_{A_i}|-|[x]_{A_i}\cap X|\le k$. So, $x \in \underline{\sum_{i=1}^{m} A_i}_{k}^{O}(X)$ that means $\underline{\sum_{i=1}^{m} A_i}_{k}^{M}(X)\subseteq\underline{\sum_{i=1}^{m} A_i}_{k}^{O}(X)$. To summarize, we can achieve that $\underline{\sum_{i=1}^{m} A_i}_{k}^{P}(X)\subseteq\underline{\sum_{i=1}^{m} A_i}_{k}^{M}(X)\subseteq\underline{\sum_{i=1}^{m} A_i}_{k}^{O}(X)$.

(2) It's similar to the process of (1).

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(3) For any $x \in \underline{\sum_{i=1}^{m} A_i}_k^M(X)$, we know $\frac{1}{m}\sum\limits_{i=1}^{m}(|[x]_{A_i}| - |[x]_{A_i} \cap X|) \le k$. Meanwhile, for any $X, Y \in \mathcal{P}(U)$ have $[x]_{A_i} \cap X \subseteq [x]_{A_i} \cap Y$ if $X \subseteq Y$. So, we can obtain that $k \ge \frac{1}{m}\sum\limits_{i=1}^{m}(|[x]_{A_i}| - |[x]_{A_i} \cap X|) \ge \frac{1}{m}\sum\limits_{i=1}^{m}(|[x]_{A_i}| - |[x]_{A_i} \cap Y|)$ means $x \in \underline{\sum_{i=1}^{m} A_i}_k^M(Y)$. That is to say $\underline{\sum_{i=1}^{m} A_i}_k^M(X) \subseteq \underline{\sum_{i=1}^{m} A_i}_k^M(Y)$ if $X \subseteq Y \in \mathcal{P}(U)$.

(4) It is easy to prove based on Definition 3.5 and the proof of (3).

(5) Let $x \in \underline{\sum_{i=1}^{m} A_i}_{k_1}^M(X)$ have $\frac{1}{m}\sum_{i=1}^{m}(|[x]_{A_i}| - |[x]_{A_i} \cap X|) \le k_1 \le k_2$ if $k_1 \le k_2$. That means $x \in \underline{\sum_{i=1}^{m} A_i}_{k_2}^M(X)$, that is to say, $\underline{\sum_{i=1}^{m} A_i}_{k_1}^M(X) \subseteq \underline{\sum_{i=1}^{m} A_i}_{k_2}^M(X)$.

(6) For any $x \in \overline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta_2)}^M(X)$ have $\frac{1}{m}\sum\limits_{i=1}^{m}(P(X|[x]_{A_i})) > \beta_2 \ge \beta_1$ if $\beta_1 \le \beta_2$. So, we can achieve that $x \in \overline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta_1)}^M(X)$, namely, $\overline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta_1)}^M(X) \supseteq \overline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta_2)}^M(X)$.

Thus, the proof is accomplished. □

Similar to the previous double-quantitative multigranulation decision-theoretic rough set, we can obtain the decision rules of Dq-MDTRS$_{\mathrm{I}}^M$ as follows:

(P_{I}^M) If $\frac{1}{m}\sum_{i=1}^{m}(|[x]_{A_i}| - |[x]_{A_i} \cap X|) \le k$ and $\frac{1}{m}\sum_{i=1}^{m}(P(X|[x]_{A_i})) > \beta$, decide $pos_{\mathrm{I}}^M(X)$.
(N_{I}^M) If $\frac{1}{m}\sum_{i=1}^{m}(|[x]_{A_i}| - |[x]_{A_i} \cap X|) > k$ and $\frac{1}{m}\sum_{i=1}^{m}(P(X|[x]_{A_i})) \le \beta$, decide $neg_{\mathrm{I}}^M(X)$.
(UB_{I}^M) If $\frac{1}{m}\sum_{i=1}^{m}(|[x]_{A_i}| - |[x]_{A_i} \cap X|) > k$ and $\frac{1}{m}\sum_{i=1}^{m}(P(X|[x]_{A_i})) > \beta$, decide $Ubn_{\mathrm{I}}^M(X)$.
(LB_{I}^M) If $\frac{1}{m}\sum_{i=1}^{m}(|[x]_{A_i}| - |[x]_{A_i} \cap X|) \le k$ and $\frac{1}{m}\sum_{i=1}^{m}(P(X|[x]_{A_i})) \le \beta$, decide $Lbn_{\mathrm{I}}^M(X)$.

There is another mean double-quantitative multigranulation decision-theoretic rough set model. It was established by combining the lower approximation operator of DTRS and upper approximation operator of GRS in multigranulation approximation space.

Definition 3.6. (Dq-MDTRS$_{\mathrm{II}}^M$) Let $I = (U, AT, V, f)$ be an information system, given $A_1, A_2, \cdots, A_m \subseteq AT$ are granular structures, for any $X \in \mathcal{P}(U)$, $\beta \le \alpha \in [0, 1]$, and $k \in \mathbf{N}$. Then, the lower and upper approximations of second kind of mean double-quantitative multigranulation decision-theoretic rough set are denoted as follows:

$$\underline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^M(X) = \{x \in U : \frac{1}{m}\sum_{i=1}^{m}(P(X|[x]_{A_i})) \ge \alpha\}, \tag{3.15}$$

$$\overline{\sum_{i=1}^{m} A_i}_k^M(X) = \{x \in U : \frac{1}{m}\sum_{i=1}^{m}(|[x]_{A_i} \cap X|) > k\}. \tag{3.16}$$

Based on the definitions of lower and upper approximations, we can characterize the rough regions by following way. They are positive region, negative region, upper boundary region an lower boundary region, respectively.

(1) $pos_{\mathrm{II}}^M(X) = \underline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^M(X) \cap \overline{\sum_{i=1}^{m} A_i}_k^M(X)$;
(2) $neg_{\mathrm{II}}^M(X) = \sim (\underline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^M(X) \cup \overline{\sum_{i=1}^{m} A_i}_k^M(X))$;
(3) $Ubn_{\mathrm{II}}^M(X) = \overline{\sum_{i=1}^{m} A_i}_k^M(X) - \underline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^M(X)$;
(4) $Lbn_{\mathrm{II}}^M(X) = \underline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^M(X) - \overline{\sum_{i=1}^{m} A_i}_k^M(X)$.

Similar to the Dq-MDTRS$_{\mathrm{I}}^M$, we can achieve the propositions of Dq-MDTRS$_{\mathrm{II}}^M$ and they are represented as follows.

Proposition 3.6. *Given $A_1, A_2, \cdots, A_m \subseteq AT$ are granular structures, for any $X \in \mathcal{P}(U)$, $\beta \le \alpha \in [0, 1]$, $k \in \mathbf{N}$. Then, the following properties hold.*

(1) $\underline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^P(X) \subseteq \underline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^M(X) \subseteq \underline{\sum_{i=1}^{m} A_i}_{(\alpha,\ \beta)}^O(X)$;
(2) $\overline{\sum_{i=1}^{m} A_i}_k^P(X) \supseteq \overline{\sum_{i=1}^{m} A_i}_k^M(X) \supseteq \overline{\sum_{i=1}^{m} A_i}_k^O(X)$;

(3) $\underline{\sum_{i=1}^{m} A_i}^{M}_{(\alpha,\ \beta)}(X) \subseteq \underline{\sum_{i=1}^{m} A_i}^{M}_{(\alpha,\ \beta)}(Y)$, *if* $X \subseteq Y \in \mathcal{P}(U)$;

(4) $\overline{\sum_{i=1}^{m} A_{i_k}}^{M}(X) \subseteq \overline{\sum_{i=1}^{m} A_{i_k}}^{M}(Y)$, *if* $X \subseteq Y \in \mathcal{P}(U)$;

(5) $\underline{\sum_{i=1}^{m} A_i}^{M}_{(\alpha_1,\ \beta)}(X) \supseteq \underline{\sum_{i=1}^{m} A_i}^{M}_{(\alpha_2,\ \beta)}(X)$, *if* $\alpha_1 \le \alpha_2 \in [0, 1]$;

(6) $\overline{\sum_{i=1}^{m} A_{i_{k_1}}}^{M}(X) \supseteq \overline{\sum_{i=1}^{m} A_{i_{k_2}}}^{M}(X)$, *if* $k_1 \le k_2 \in \mathbf{N}$.

Proof. It is similar to the Proposition 3.5. □

The Dq-MDTRS$_{\text{II}}^{M}$ is also a generalization of pessimistic multigranulation rough set model. It indicates that the Dq-MDTRS$_{\text{II}}^{M}$ will be degenerated into a pessimistic multigranulation rough set model when $\alpha = 1$ and $k = 0$. The Dq-MDTRS$_{\text{II}}^{M}$ is a double-quantitative decision-theoretic rough set model which consists relative and absolute quantification of information. The decision rules of Dq-MDTRS$_{\text{II}}^{M}$ are represented as follows:

(P_{II}^{M}) If $\frac{1}{m}\sum_{i=1}^{m}(P(X|[x]_{A_i})) \ge \alpha$ and $\frac{1}{m}\sum_{i=1}^{m}(|[x]_{A_i}| - |[x]_{A_i} \cap X|) > k$, decide $pos_{\text{II}}^{M}(X)$.
(N_{II}^{M}) If $\frac{1}{m}\sum_{i=1}^{m}(P(X|[x]_{A_i})) < \alpha$ and $\frac{1}{m}\sum_{i=1}^{m}(|[x]_{A_i}| - |[x]_{A_i} \cap X|) \le k$, decide $neg_{\text{II}}^{M}(X)$.
(UB_{II}^{M}) If $\frac{1}{m}\sum_{i=1}^{m}(P(X|[x]_{A_i})) < \alpha$ and $\frac{1}{m}\sum_{i=1}^{m}(|[x]_{A_i}| - |[x]_{A_i} \cap X|) > k$, decide $Ubn_{\text{II}}^{M}(X)$.
(LB_{II}^{M}) If $\frac{1}{m}\sum_{i=1}^{m}(P(X|[x]_{A_i})) \ge \alpha$ and $\frac{1}{m}\sum_{i=1}^{m}(|[x]_{A_i}| - |[x]_{A_i} \cap X|) \le k$, decide $Lbn_{\text{II}}^{M}(X)$.

Thus, we established six double-quantitative multigranulation decision-theoretic rough set (Dq-MGDTRS) models which consist of Dq-MDTRS$_{\text{I}}^{O}$, Dq-MDTRS$_{\text{II}}^{O}$, Dq-MDTRS$_{\text{I}}^{P}$, Dq-MDTRS$_{\text{II}}^{P}$, Dq-MDTRS$_{\text{I}}^{M}$ and Dq-MDTRS$_{\text{II}}^{M}$, respectively. The mathematical properties of these models are investigated and the decision rules of models are represented. In next subsection, we will try to discuss the relationships between Dq-MGDTRS and other RS models.

3.5. Relationships between Dq-MGDTRS and other RS models

Based on previous discussions, we obtain six Dq-MGDTRS models in this study. In these models, the conditional probability value (thresholds α and β) and grade (threshold k) decide their detailed form of rough set. From Yao's study [54], we can get that the decision-theoretic rough set (DTRS) is a special case of probabilistic rough set (PRS, in [25]) and the DTRS will degenerate into the Pawlak rough set if thresholds $\alpha = 1$ and $\beta = 0$. So, we have the following formulas according to the formula (2.5) in this scenario.

(1) For $\alpha = 1$, we can get that $P(X|[x]_A) = |[x]_A \cap X|/|[x]_A| = 1$ means $|[x]_A \cap X| = |[x]_A|$, that is $[x]_A \subseteq X$.
(2) For $\beta = 0$, we can obtain that $P(X|[x]_A) = |[x]_A \cap X|/|[x]_A| = 0$, namely, $|[x]_A \cap X| = 0$, that is $[x]_A \cap X = \emptyset$.
Therefore, the formulas (2.11) and (2.12) can be presented as follows:

$$\underline{A}_{(1,\ 0)}(X) = \{x \in U : P(X|[x]_A) \ge 1\} = \{x \in U : [x]_A \subseteq X\},$$
$$\overline{A}_{(1,\ 0)}(X) = \{x \in U : P(X|[x]_A) > 0\} = \{x \in U : [x]_A \cap X \ne \emptyset\}.$$

According to [50], the graded rough set will degenerate into the Pawlak rough set if threshold $k = 0$. Here, we can get the descriptions of internal and external grades as follows:
(3) For $k = 0$, we can get that $|[x]_A| - |[x]_A \cap X| = 0$ means $|[x]_A| = |[x]_R \cap X|$ that is $[x]_A \subseteq X$.
(4) For $k = 0$, we can obtain that $|[x]_A \cap X| = 0$, namely, $[x]_A \cap X = \emptyset$.
Hence, the formulas (2.13) and (2.14) can be characterized as follows:

$$\underline{A}_{k=0}(X) = \{x \in U : |[x]_A| - |[x]_A \cap X| \le 0\} = \{x \in U : [x]_A \subseteq X\},$$
$$\overline{A}_{k=0}(X) = \{x \in U : |[x]_A \cap X| > 0\} = \{x \in U : [x]_A \cap X \ne \emptyset\}.$$

Accordingly, we can obtain that both the DTRS and GRS are generalization of Pawlak RS. Especially, the DTRS and GRS are equivalent if $\alpha = 1$, $\beta = 0$ and $k = 0$. For the double-quantification topic, Zhang developed an double-quantitative rough set (Dq-RS) model regarding probabilities and grades in [65], and Li proposed an double-quantitative decision-theoretic rough set (Dq-DTRS) model based on Bayesian decision procedure and GRS [13]. On the other hand, the MGRS is a natural prolongation of Pawlak RS when multi binary relations should be considered [31]. It is not difficult for us to infer that Pawlak RS is a degeneration of MGRS. Furthermore, the multigranulation decision-theoretic rough set (MGDTRS) is established by utilizing the decision-theoretic approach to multigranulation approximate space. Based on these models, this talk discussed the double-quantitative multigranulation decision-theoretic rough set (Dq-MGDTRS) models. Therefore, the relationships between the Dq-MGDTRS and other models are exhibited in Fig. 1.

According to the above discussions, we can obtain that the Dq-MGDTRS is an expansion of Dq-DTRS and MGDTRS. In particular, the Dq-MGDTRS will degenerate into MGRS if $\alpha = 1$, $\beta = 0$ and $k = 0$. That means the Dq-MDTRS$_{\text{I}}^{O}$ and

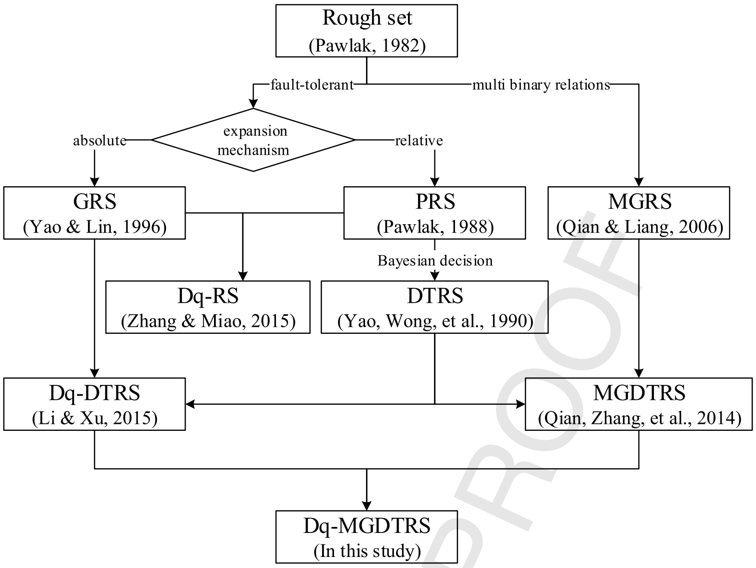

Fig. 1. Relationship between Dq-MGDTRS and other RS models.

Dq-MDTRS$_{\mathrm{II}}^{O}$ will degenerate into optimistic multigranulation rough set, the Dq-MDTRS$_{\mathrm{I}}^{P}$ and Dq-MDTRS$_{\mathrm{II}}^{P}$ will degenerate into pessimistic multigranulation rough set. For the Dq-MDTRS$_{\mathrm{I}}^{M}$ and Dq-MDTRS$_{\mathrm{II}}^{M}$, we conduct the following discussions. Based on Definition 3.5, we can obtain the following formulas when $\alpha = 1$, $\beta = 0$ and $k = 0$.

$$\underline{\sum_{i=1}^{m} A_i}^{M}_{k=0}(X) = \{x \in U : \frac{1}{m}\sum_{i=1}^{m}(|[x]_{A_i}| - |[x]_{A_i} \cap X|) \le k\} = \{x \in U : \bigwedge_{i=1}^{m}([x]_{A_i} \subseteq X)\},$$

$$\overline{\sum_{i=1}^{m} A_i}^{M}_{(1,\ 0)}(X) = \{x \in U : \frac{1}{m}\sum_{i=1}^{m}(P(X|[x]_{A_i})) > \beta\} = \{x \in U : \bigvee_{i=1}^{m}([x]_{A_i} \cap X \ne \emptyset)\}.$$

Proof. Since for any $x \in U$ and $A_i \subseteq AT$ $(i = 1, 2, \cdots, m)$, we have $|[x]_{A_i}| - |[x]_{A_i} \cap X| \ge 0$. So, we can get $|[x]_{A_i}| - |[x]_{A_i} \cap X| = 0$ if $k = 0$. That is $|[x]_{A_i}| = |[x]_{A_i} \cap X|$, namely, $[x]_{A_i} \subseteq X$ hold for all $A_i \subseteq AT$. Therefore, $\underline{\sum_{i=1}^{m} A_i}^{M}_{k=0}(X) = \{x \in U : \bigwedge_{i=1}^{m}([x]_{A_i} \subseteq X)\}$. On the other hand, because $P(X|[x]_{A_i}) \ge 0$ for any $A_i \subseteq AT$ $(i = 1, 2, \cdots, m)$. Here, we can obtain $\frac{1}{m}\sum_{i=1}^{m}(P(X|[x]_{A_i})) > 0$ when $\beta = 0$. That means there exists one $A_i \subseteq AT$ such that $P(X|[x]_{A_i}) > 0$, that is, $\overline{\sum_{i=1}^{m} A_i}^{M}_{(1,\ 0)}(X) = \{x \in U : \bigvee_{i=1}^{m}([x]_{A_i} \cap X \ne \emptyset)\}$. This completes the proof. □

Therefore, we can obtain that the Dq-MDTRS$_{\mathrm{I}}^{M}$ is an expansion of pessimistic multigranulation rough set. Analogously, we can get that the Dq-MDTRS$_{\mathrm{II}}^{M}$ will also degenerate into pessimistic multigranulation rough set if $\alpha = 1$, $\beta = 0$ and $k = 0$. It can be proved by a similar method. They indicate that these models with their thresholds exhibit favorable directional expansion properties. Based on the above discussions, we will conduct a case study in next section.

4. Case study

Compared with the classical multigranulation decision-theoretic rough set model, the double-quantitative multigranulation decision-theoretic rough set models consider not only the relative quantitative information but also the absolute quantitative information between the indiscernibility classes and concept set. With the application of thresholds k, α and β the fault-tolerant ability of model is improved. In order to exhibit the decision approach that combining relative and absolute quantitative simultaneously, we conduct an example based on Zhang's investigation [62]. These experiments are implemented using Matlab R2014a and performed on a personal computer with an Intel Core i3-4150, 3.50 GHz CPU, 4.0 GB of memory, and 64-bit Windows 7 OS.

Let $I = (U, AT, V, f)$ be an information system with multiple granular structures, where U is composed of 36 patients, and the condition and decision attributes are fever, headache, cough and cold respectively. They are represented as a_1, a_2, a_3 and d in the following discussion. The detailed characteristics of the datasets are showed in Table 2.

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Table 2
Initial medical data.

U	a_1	a_2	a_3	d	U	a_1	a_2	a_3	d	U	a_1	a_2	a_3	d
x_1	0	0	0	0	x_{13}	0	0	1	0	x_{25}	0	2	0	0
x_2	1	1	1	0	x_{14}	2	1	2	1	x_{26}	2	2	2	1
x_3	0	2	1	1	x_{15}	0	1	2	1	x_{27}	1	1	0	0
x_4	2	1	2	0	x_{16}	1	1	0	0	x_{28}	2	0	1	1
x_5	1	0	1	1	x_{17}	0	2	1	0	x_{29}	2	1	2	1
x_6	2	2	2	1	x_{18}	2	1	2	1	x_{30}	0	0	2	0
x_7	0	0	0	0	x_{19}	0	0	0	0	x_{31}	1	2	1	0
x_8	1	2	1	0	x_{20}	1	2	2	1	x_{32}	0	1	0	0
x_9	2	2	2	1	x_{21}	2	0	1	1	x_{33}	2	1	1	1
x_{10}	1	1	1	1	x_{22}	0	0	0	0	x_{34}	1	1	1	1
x_{11}	1	2	1	1	x_{23}	2	1	0	0	x_{35}	0	0	0	0
x_{12}	2	0	0	0	x_{24}	1	2	2	1	x_{36}	2	0	1	0

For simplicity and without loss of generality, suppose there are three granular structures that $A_1 = \{a_1, a_2\}$, $A_2 = \{a_1, a_3\}$ and $A_3 = \{a_2, a_3\}$, respectively. So, we can compute the equivalence classes with respect to A_1, A_2 and A_3. From Table 2, we can get that the universe of discourse is divided to two parts that $U/d = \{D_1, D_2\}$ as follows:

$$D_1 = \{x_1, x_2, x_4, x_7, x_8, x_{12}, x_{13}, x_{16}, x_{17}, x_{19}, x_{22}, x_{23}, x_{25}, x_{27}, x_{30}, x_{31}, x_{32}, x_{35}, x_{36}\},$$

$$D_2 = \{x_3, x_5, x_6, x_9, x_{10}, x_{11}, x_{14}, x_{15}, x_{18}, x_{20}, x_{21}, x_{24}, x_{26}, x_{28}, x_{29}, x_{33}, x_{34}.$$

Where D_1 and D_2 stand for $d = 0$ and $d = 1$, while D_2 expresses the set of patients who are suffering from cold. Let concept set $X = D_2$, then we can compute these variables to characterize the double-quantitative decision-theoretic approach, which are listed in Table 3. The pair (a, b) is a sequence of attribute value with respect to one granular structure where $a, b \in \{0, 1, 2\}$. Here, it should be noted that some abbreviations are utilized to make the table looks more concise. The $\overline{g_{A_i}}$, $\underline{g_{A_i}}$ and P_{A_i} represent internal grade, external grade and conditional probability with respect to A_i, respectively. Meanwhile, the mark $x_{i,j,k}$ means a set $\{x_i, x_j, x_k\}$ throughout this paper, for instance, the $x_{1,7,13,19,22,30,35}$ expresses a set of objects that $\{x_1, x_7, x_{13}, x_{19}, x_{22}, x_{30}, x_{35}\}$.

The definition of upper approximation of DTRS indicates that $\overline{A}_{(\alpha,\ \beta)}(X)$ is purely β dependent. It has nothing to do with the α in form or appearance, but the potential condition should be satisfied that $\alpha > \beta$. So, we can compute the upper and lower approximations with respect to different thresholds, respectively. The Table 4 represents the upper approximation of X with respect to parameter β for each granular structure A_i in DTRS.

Corresponding to Table 4, Table 5 is utilized to describe the lower approximations of X with respect to α for each granular structure A_i in DTRS. In the Bayesian decision procedure, the decision-making is based on a pair of threshold $(\alpha,\ \beta)$. In general, it is divided into three cases that $\alpha + \beta > 1$, $\alpha + \beta = 1$ and $\alpha + \beta < 1$, respectively. In the calculation process, we can combine the Table 4 and Table 5 to achieve the results for an arbitrary thresholds pair $(\alpha,\ \beta)$. Then, we will discuss the decision rules based on different combinations of parameters.

According to the achievements of Table 3, we can obtain the lower and upper approximations of X with respect to different grade k for each granular structure A_i as shown in Table 6. There should be noted that in order to make the table looks more concise we use the equivalence classes to instead of elements in the description of lower and upper approximations.

For convenience and without loss of generality, we choose the grade $k = 2$ throughout this case study. Then, we can achieve the optimistic, pessimistic and mean graded multigranulation upper and lower approximations of X with respect to the grade $k = 2$, respectively. They are exhibited as follows:

$\overline{\sum_{i=1}^{m} A_{i_2}}^{O}(X) = \{x_4, x_6, x_9, x_{14}, x_{18}, x_{26}, x_{29}, x_{33}\}$,

$\underline{\sum_{i=1}^{m} A_{i_2}}^{O}(X) = U - \{x_1, x_7, x_{19}, x_{22}, x_{35}\}$.

$\overline{\sum_{i=1}^{m} A_{i_2}}^{P}(X) = \{x_4, x_6, x_9, x_{14}, x_{15}, x_{18}, x_{20}, x_{21}, x_{24}, x_{26}, x_{28}, x_{29}, x_{33}, x_{36}\}$,

$\underline{\sum_{i=1}^{m} A_{i_2}}^{P}(X) = \{x_2, x_4, x_5, x_6, x_8, x_9, x_{10}, x_{11}, x_{13}, x_{14}, x_{15}, x_{18}, x_{20}, x_{21}, x_{23}, x_{24}, x_{26}, x_{28}, x_{29}, x_{31}, x_{33}, x_{34}, x_{36}\}$.

$\overline{\sum_{i=1}^{m} A_{i_2}}^{M}(X) = \{x_2, x_4, x_5, x_6, x_8, x_9, x_{10}, x_{11}, x_{14}, x_{18}, x_{20}, x_{21}, x_{24}, x_{26}, x_{28}, x_{29}, x_{31}, x_{33}, x_{34}, x_{36}\}$,

$\underline{\sum_{i=1}^{m} A_{i_2}}^{M}(X) = \{x_4, x_5, x_6, x_9, x_{14}, x_{15}, x_{18}, x_{20}, x_{21}, x_{24}, x_{26}, x_{28}, x_{29}, x_{33}, x_{36}\}$.

In the Bayesian decision procedure, from the losses, one can give the values $\lambda_{i1}, \lambda_{i2}$ where $i = 1, 2, 3$. We can make some changes to the loss function which is defined in our prior study [6], and the thresholds can be calculated as follows:

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Table 3
Statistical results with respect to the A_i.

(a,b)	A_1					A_2					A_3				
	$[x]_{A_1}$	$[x]_{A_1} \cap X$	$\overline{g_{A_1}}$	$\underline{g_{A_1}}$	P_{A_1}	$[x]_{A_2}$	$[x]_{A_2} \cap X$	$\overline{g_{A_2}}$	$\underline{g_{A_2}}$	P_{A_2}	$[x]_{A_3}$	$[x]_{A_3} \cap X$	$\overline{g_{A_3}}$	$\underline{g_{A_3}}$	P_{A_3}
(0,0)	$x_{1,7,13,19,22,30,35}$	$\emptyset$	0	7	0	$x_{1,7,19,22,25,32,35}$	$\emptyset$	0	7	0	$x_{1,7,12,19,22,35}$	$\emptyset$	0	6	0
(0,1)	$x_{15,32}$	x_{15}	1	1	1/2	$x_{3,13,17}$	x_3	1	2	1/3	$x_{5,13,21,28,36}$	$x_{5,21,28}$	3	2	3/5
(0,2)	$x_{3,17,25}$	x_3	1	2	1/3	$x_{15,30}$	x_{15}	1	1	1/2	x_{30}	$\emptyset$	0	1	0
(1,0)	x_5	x_5	1	0	1	$x_{16,27}$	$\emptyset$	0	2	0	$x_{16,23,27,32}$	$\emptyset$	0	4	0
(1,1)	$x_{2,10,16,27,34}$	$x_{10,34}$	2	3	2/5	$x_{2,5,8,10,11,31,34}$	$x_{5,10,11,34}$	4	3	4/7	$x_{2,10,33,34}$	$x_{10,33,34}$	3	1	3/4
(1,2)	$x_{8,11,20,24,31}$	$x_{11,20,24}$	3	2	3/5	$x_{20,24}$	$x_{20,24}$	2	0	1	$x_{4,14,15,18,29}$	$x_{14,15,18,29}$	4	1	4/5
(2,0)	$x_{12,21,28,36}$	$x_{21,28}$	2	2	1/2	$x_{12,23}$	$\emptyset$	0	2	0	x_{25}	$\emptyset$	0	1	0
(2,1)	$x_{4,14,18,29,33}$	$x_{14.18,23,29,33}$	4	2	2/3	$x_{21,28,33,36}$	$x_{21,28,33}$	3	1	3/4	$x_{3,8,11,17,31}$	$x_{3,11}$	2	3	2/5
(2,2)	$x_{6,9,26}$	$x_{6,9,26}$	3	0	1	$x_{4,6,9,14,18,26,29}$	$x_{6,9,14,18,26,29}$	6	1	6/7	$x_{6,9,20,24,26}$	$x_{6,9,20,24,26}$	5	0	1

Table 4
Upper approximation regarding A_i in DTRS.

β	A_1	A_2	A_3
0.3	$U-[x_1]_{A_1}$	$U-[x_1]_{A_2}-[x_{16}]_{A_2}-[x_{12}]_{A_2}$	$[x_5]_{A_3}\cup[x_2]_{A_3}\cup[x_4]_{A_3}\cup[x_3]_{A_3}\cup[x_6]_{A_3}$
0.4	$U-[x_1]_{A_1}-[x_3]_{A_1}-[x_2]_{A_1}$	$[x_{15}]_{A_2}\cup[x_2]_{A_2}\cup[x_{20}]_{A_2}\cup[x_{21}]_{A_2}\cup[x_4]_{A_2}$	$[x_5]_{A_3}\cup[x_2]_{A_3}\cup[x_4]_{A_3}\cup[x_6]_{A_3}$
0.5	$[x_5]_{A_1}\cup[x_8]_{A_1}\cup[x_4]_{A_1}\cup[x_6]_{A_1}$	$[x_2]_{A_2}\cup[x_{20}]_{A_2}\cup[x_{21}]_{A_2}\cup[x_4]_{A_2}$	$[x_5]_{A_3}\cup[x_2]_{A_3}\cup[x_4]_{A_3}\cup[x_6]_{A_3}$
0.6	$[x_5]_{A_1}\cup[x_4]_{A_1}\cup[x_6]_{A_1}$	$[x_{20}]_{A_2}\cup[x_{21}]_{A_2}\cup[x_4]_{A_2}$	$[x_2]_{A_3}\cup[x_4]_{A_3}\cup[x_6]_{A_3}$
0.7	$[x_5]_{A_1}\cup[x_4]_{A_1}\cup[x_6]_{A_1}$	$[x_{20}]_{A_2}\cup[x_{21}]_{A_2}\cup[x_4]_{A_2}$	$[x_2]_{A_3}\cup[x_4]_{A_3}\cup[x_6]_{A_3}$
0.8	$[x_5]_{A_1}\cup[x_4]_{A_1}\cup[x_6]_{A_1}$	$[x_{20}]_{A_2}\cup[x_4]_{A_2}$	$[x_6]_{A_3}$
0.9	$[x_5]_{A_1}\cup[x_6]_{A_1}$	$[x_{20}]_{A_2}$	$[x_6]_{A_3}$

Table 5
Lower approximation regarding A_i in DTRS.

α	A_1	A_2	A_3
0.3	$U-[x_1]_{A_1}$	$U-[x_1]_{A_2}-[x_{16}]_{A_2}-[x_{12}]_{A_2}$	$[x_5]_{A_3}\cup[x_2]_{A_3}\cup[x_4]_{A_3}\cup[x_3]_{A_3}\cup[x_6]_{A_3}$
0.4	$U-[x_1]_{A_1}-[x_3]_{A_1}$	$[x_{15}]_{A_2}\cup[x_2]_{A_2}\cup[x_{20}]_{A_2}\cup[x_{21}]_{A_2}\cup[x_4]_{A_2}$	$[x_5]_{A_3}\cup[x_2]_{A_3}\cup[x_4]_{A_3}\cup[x_3]_{A_3}\cup[x_6]_{A_3}$
0.5	$U-[x_1]_{A_1}-[x_3]_{A_1}-[x_2]_{A_1}$	$[x_{15}]_{A_2}\cup[x_2]_{A_2}\cup[x_{20}]_{A_2}\cup[x_{21}]_{A_2}\cup[x_4]_{A_2}$	$[x_5]_{A_3}\cup[x_2]_{A_3}\cup[x_4]_{A_3}\cup[x_6]_{A_3}$
0.6	$[x_5]_{A_1}\cup[x_8]_{A_1}\cup[x_4]_{A_1}\cup[x_6]_{A_1}$	$[x_{20}]_{A_2}\cup[x_{21}]_{A_2}\cup[x_4]_{A_2}$	$[x_5]_{A_3}\cup[x_2]_{A_3}\cup[x_4]_{A_3}\cup[x_6]_{A_3}$
0.7	$[x_5]_{A_1}\cup[x_4]_{A_1}\cup[x_6]_{A_1}$	$[x_{20}]_{A_2}\cup[x_{21}]_{A_2}\cup[x_4]_{A_2}$	$[x_2]_{A_3}\cup[x_4]_{A_3}\cup[x_6]_{A_3}$
0.8	$[x_5]_{A_1}\cup[x_4]_{A_1}\cup[x_6]_{A_1}$	$[x_{20}]_{A_2}\cup[x_4]_{A_2}$	$[x_4]_{A_3}\cup[x_6]_{A_3}$
0.9	$[x_5]_{A_1}\cup[x_6]_{A_1}$	$[x_{20}]_{A_2}$	$[x_6]_{A_3}$
1.0	$[x_5]_{A_1}\cup[x_6]_{A_1}$	$[x_{20}]_{A_2}$	$[x_6]_{A_3}$

Table 6
Upper and lower approximations regarding A_i in GRS.

k	App.	A_1	A_2	A_3
0	$\overline{R}_k$	$U-[x_1]_{A_1}$	$U-[x_1]_{A_2}-[x_{16}]_{A_2}-[x_{16}]_{A_{12}}$	$[x_5]_{A_3}\cup[x_2]_{A_3}\cup[x_4]_{A_3}\cup[x_3]_{A_3}\cup[x_6]_{A_3}$
	$\underline{R}_k$	$[x_5]_{A_1}\cup[x_6]_{A_1}$	$[x_{20}]_{A_2}$	$[x_6]_{A_3}$
1	$\overline{R}_k$	$[x_2]_{A_1}\cup[x_8]_{A_1}\cup[x_{12}]_{A_1}\cup[x_4]_{A_1}\cup[x_6]_{A_1}$	$[x_2]_{A_2}\cup[x_{20}]_{A_2}\cup[x_{21}]_{A_2}\cup[x_4]_{A_2}$	$[x_5]_{A_3}\cup[x_2]_{A_3}\cup[x_4]_{A_3}\cup[x_3]_{A_3}\cup[x_6]_{A_3}$
	$\underline{R}_k$	$[x_{15}]_{A_1}\cup[x_5]_{A_1}\cup[x_4]_{A_1}\cup[x_6]_{A_1}$	$[x_{15}]_{A_2}\cup[x_{20}]_{A_2}\cup[x_{21}]_{A_2}\cup[x_4]_{A_2}$	$[x_{30}]_{A_3}\cup[x_2]_{A_3}\cup[x_4]_{A_3}\cup[x_{25}]_{A_3}\cup[x_6]_{A_3}$
2	$\overline{R}_k$	$[x_8]_{A_1}\cup[x_4]_{A_1}\cup[x_6]_{A_1}$	$[x_2]_{A_2}\cup[x_{21}]_{A_2}\cup[x_4]_{A_2}$	$[x_5]_{A_3}\cup[x_2]_{A_3}\cup[x_4]_{A_3}\cup[x_6]_{A_3}$
	$\underline{R}_k$	$U-[x_1]_{A_1}-[x_2]_{A_1}$	$U-[x_1]_{A_2}-[x_2]_{A_2}$	$U-[x_1]_{A_3}-[x_{16}]_{A_3}-[x_3]_{A_3}$
3	$\overline{R}_k$	$[x_4]_{A_1}$	$[x_2]_{A_2}\cup[x_4]_{A_2}$	$[x_4]_{A_3}\cup[x_6]_{A_3}$
	$\underline{R}_k$	$U-[x_1]_{A_1}-[x_2]_{A_1}$	$U-[x_1]_{A_2}$	$U-[x_1]_{A_3}-[x_{16}]_{A_3}$
4	$\overline{R}_k$	$[x_4]_{A_1}$	$[x_4]_{A_2}$	$[x_6]_{A_3}$
	$\underline{R}_k$	$U-[x_1]_{A_1}$	$U-[x_1]_{A_2}$	$U-[x_1]_{A_3}$
5	$\overline{R}_k$	$\emptyset$	$[x_4]_{A_2}$	$\emptyset$
	$\underline{R}_k$	$U-[x_1]_{A_1}$	$U-[x_1]_{A_2}$	$U-[x_1]_{A_3}$

Case 1. $\alpha+\beta=1$. Consider the following loss function:

$$\lambda_{PP}=0,\ \ \lambda_{PN}=18,$$
$$\lambda_{BP}=9,\ \ \ \lambda_{BN}=2,$$
$$\lambda_{NP}=12,\ \ \lambda_{NN}=0.$$

Then, we can get $\alpha=0.6$, $\beta=0.4$ that means $\alpha+\beta=1$. We can achieve the optimistic, pessimistic and mean multi-granulation decision-theoretic upper and lower approximations of concept X, respectively.

$$\overline{\sum_{i=1}^{m} A_i}^{O}_{(0.6,\ 0.4)}(X)=\{x_4,x_5,x_6,x_9,x_{14},x_{15},x_{18},x_{20},x_{21},x_{24},x_{26},x_{28},x_{29},x_{33},x_{36}\},$$

$$\underline{\sum_{i=1}^{m} A_i}^{O}_{(0.6,\ 0.4)}(X)=\{x_2,x_4,x_5,x_6,x_8,x_9,x_{10},x_{11},x_{13},x_{14},x_{15},x_{18},x_{20},x_{21},x_{23},x_{24},x_{26},x_{28},x_{29},x_{31},x_{33},x_{34},x_{36}\}.$$

$$\overline{\sum_{i=1}^{m} A_i}^{P}_{(0.6,\ 0.4)}(X)=U-\{x_1,x_3,x_7,x_{16},x_{17},x_{19},x_{22},x_{25},x_{27},x_{35}\},$$

$$\underline{\sum_{i=1}^{m} A_i}^{P}_{(0.6,\ 0.4)}(X)=\{x_4,x_6,x_9,x_{14},x_{18},x_{20},x_{24},x_{26},x_{29},x_{33}\}.$$

$$\overline{\sum_{i=1}^{m} A_i}^{M}_{(0.6,\ 0.4)}(X)=\{x_2,x_4,x_5,x_6,x_8,x_9,x_{10},x_{11},x_{14},x_{15},x_{18},x_{20},x_{21},x_{24},x_{26},x_{28},x_{29},x_{31},x_{33},x_{34},x_{36}\},$$

$$\underline{\sum_{i=1}^{m} A_i}^{M}_{(0.6,\ 0.4)}(X)=\{x_4,x_5,x_6,x_9,x_{14},x_{15},x_{18},x_{20},x_{21},x_{24},x_{26},x_{28},x_{29},x_{33},x_{36}\}.$$

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Table 7
The Dq-MDT approximations of X with respect to $k=2$, $\alpha=0.6$ and $\beta=0.4$.

Model	Lower approximation	Upper approximation
Dq-MDTRS$_{\mathrm{I}}^{O}$	$U - x_{1,7,19,22,35}$	$x_{4,5,6,9,14,15,18,20,21,24,26,28,29,33,36}$
Dq-MDTRS$_{\mathrm{II}}^{O}$	$x_{2,4,5,6,8,9,10,11,13,14,15,18,20,21,23,24,26,28,29,31,33,34,36}$	$x_{4,6,9,14,18,26,29,33}$
Dq-MDTRS$_{\mathrm{I}}^{P}$	$x_{4,6,9,14,15,18,20,21,24,26,28,29,33,36}$	$x_{2,4,5,6,8,9,10,11,12,13,14,15,18,20,21,23,24,26,28,29,30,31,32,33,34,36}$
Dq-MDTRS$_{\mathrm{II}}^{P}$	$x_{4,6,9,14,18,20,24,26,29,33}$	$x_{2,4,5,6,8,9,10,11,13,14,15,18,20,21,23,24,26,28,29,31,33,34,36}$
Dq-MDTRS$_{\mathrm{I}}^{M}$	$x_{4,5,6,9,14,15,18,20,21,24,26,28,29,33,36}$	$x_{2,4,5,6,8,9,10,11,14,15,18,20,21,24,26,28,29,31,33,34,36}$
Dq-MDTRS$_{\mathrm{II}}^{M}$	$x_{4,5,6,9,14,15,18,20,21,24,26,28,29,33,36}$	$x_{2,4,5,6,8,9,10,11,14,18,20,21,24,26,28,29,31,33,34,36}$

Based on these results and the approximations of X with respect to $k=2$ in GRS, we can obtain the lower and upper approximations of X for the proposed double-quantitative multigranulation decision-theoretic rough set models, respectively. They are shown in Table 7.

According to Table 7, we can compute the rough regions of Dq-MDTRS$_{\mathrm{I}}^{O}$, Dq-MDTRS$_{\mathrm{II}}^{O}$, Dq-MDTRS$_{\mathrm{I}}^{P}$, Dq-MDTRS$_{\mathrm{II}}^{P}$, Dq-MDTRS$_{\mathrm{I}}^{M}$ and Dq-MDTRS$_{\mathrm{II}}^{M}$ with respect to $\alpha=0.6$, $\beta=0.4$ and $k=2$, respectively. For simplicity and without loss of generality, we compute the rough regions of Dq-MDTRS$_{\mathrm{I}}^{O}$ as an illustration, and they are listed as follows:

$pos_{\mathrm{I}}^{O}(X) = \{x_4, x_5, x_6, x_9, x_{14}, x_{15}, x_{18}, x_{20}, x_{21}, x_{24}, x_{26}, x_{28}, x_{29}, x_{33}, x_{36}\}$;

$neg_{\mathrm{I}}^{O}(X) = \{x_1, x_7, x_{19}, x_{22}, x_{35}\}$;

$Ubn_{\mathrm{I}}^{O}(X) = \emptyset$;

$Lbn_{\mathrm{I}}^{O}(X) = \{x_2, x_3, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{16}, x_{17}, x_{23}, x_{25}, x_{27}, x_{30}, x_{31}, x_{32}, x_{34}\}$.

For $\alpha=0.6$, $\beta=0.4$ and $k=2$, these models have their own quantitative semantics for the relative and absolute degree quantification. Furthermore, we can obtain the decision rules in practiced applications by using Dq-MDTRS$_{\mathrm{I}}^{O}$ model as follows:

(P_{I}^{O}) The patients $x_3, x_4, x_6, x_8, x_9, x_{11}, x_{12}, x_{14}, x_{17}, x_{18}, x_{20}, x_{21}, x_{23}, x_{24}, x_{25}, x_{26}, x_{28}, x_{29}, x_{31}, x_{33}$ and x_{36} are suffering from cold with respect to these diagnostic indexes and given parameters;

(N_{I}^{O}) The patients x_1, x_7, x_{19}, x_{22} and x_{35} are not suffering from cold regarding current diagnostic conditions;

(B_{I}^{O}) The patients $x_2, x_3, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{16}, x_{17}, x_{23}, x_{25}, x_{27}, x_{30}, x_{31}, x_{32}$ and x_{34} can not be diagnosed with respect to present information. A further diagnosis is need to them.

Case 2. $\alpha+\beta<1$. Consider the following loss function:

$$\lambda_{PP}=0,\ \ \lambda_{PN}=19,$$
$$\lambda_{BP}=12,\ \ \lambda_{BN}=3,$$
$$\lambda_{NP}=19,\ \ \lambda_{NN}=0.$$

Based on the loss function, we can get $\alpha=0.5$, $\beta=0.3$, that is to say, $\alpha+\beta<1$. We can obtain the optimistic, pessimistic and mean multigranulation decision-theoretic upper and lower approximations of concept X, respectively.

$\overline{\sum_{i=1}^{m} A_i}^{O}_{(0.5,\ 0.3)}(X) = \{x_2, x_3, x_4, x_5, x_6, x_8, x_9, x_{10}, x_{11}, x_{14}, x_{15}, x_{17}, x_{18}, x_{20}, x_{21}, x_{24}, x_{26}, x_{28}, x_{29}, x_{31}, x_{33}, x_{34}, x_{36}\}$,

$\underline{\sum_{i=1}^{m} A_i}^{O}_{(0.5,\ 0.3)}(X) = U - \{x_1, x_3, x_7, x_{16}, x_{17}, x_{19}, x_{22}, x_{25}, x_{27}, x_{35}\}$.

$\overline{\sum_{i=1}^{m} A_i}^{P}_{(0.5,\ 0.3)}(X) = U - \{x_1, x_7, x_{19}, x_{22}, x_{35}\}$,

$\underline{\sum_{i=1}^{m} A_i}^{P}_{(0.5,\ 0.3)}(X) = \{x_4, x_5, x_6, x_9, x_{14}, x_{15}, x_{18}, x_{20}, x_{21}, x_{24}, x_{26}, x_{28}, x_{29}, x_{33}, x_{36}\}$.

$\overline{\sum_{i=1}^{m} A_i}^{M}_{(0.5,\ 0.3)}(X) = \{x_2, x_3, x_4, x_5, x_6, x_8, x_9, x_{10}, x_{11}, x_{13}, x_{14}, x_{15}, x_{17}, x_{18}, x_{20}, x_{21}, x_{24}, x_{26}, x_{28}, x_{29}, x_{31}, x_{33}, x_{34}, x_{36}\}$,

$\underline{\sum_{i=1}^{m} A_i}^{M}_{(0.5,\ 0.3)}(X) = \{x_2, x_4, x_5, x_6, x_8, x_9, x_{10}, x_{11}, x_{14}, x_{15}, x_{18}, x_{20}, x_{21}, x_{24}, x_{26}, x_{28}, x_{29}, x_{31}, x_{33}, x_{34}, x_{36}\}$.

Combining with these results and the achievements that are showed in Table 6, we can obtain the lower and upper approximations of these constructed double-quantitative multigranulation decision-theoretic rough set models and they are exhibited in Table 8.

Table 8 indicates that the lower and upper approximations of Dq-MDTRS$_{\mathrm{I}}^{O}$, Dq-MDTRS$_{\mathrm{II}}^{O}$, Dq-MDTRS$_{\mathrm{I}}^{P}$, Dq-MDTRS$_{\mathrm{II}}^{P}$, Dq-MDTRS$_{\mathrm{I}}^{M}$ and Dq-MDTRS$_{\mathrm{II}}^{M}$ with respect to $\alpha=0.5$, $\beta=0.3$ and $k=2$, respectively. Based on these results, we can directly obtain the rough regions by the definitions. For simplicity and without loss of generality, we choose the Dq-MDTRS$_{\mathrm{I}}^{O}$ as an example, and the rough regions are exhibited by following way.

$pos_{\mathrm{I}}^{P}(X) = \{x_4, x_6, x_9, x_{14}, x_{15}, x_{18}, x_{20}, x_{21}, x_{24}, x_{26}, x_{28}, x_{29}, x_{33}, x_{36}\}$;

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Table 8
The Dq-MDT approximations of X with respect to $k = 2$, $\alpha = 0.5$ and $\beta = 0.3$.

Model	Lower approximation	Upper approximation
Dq-MDTRS$_{\mathrm{I}}^{O}$	$U - x_{1,7,19,22,35}$	$x_{2,3,4,5,6,8,9,10,11,14,15,17,18,20,21,24,26,28,29,31,33,34,36}$
Dq-MDTRS$_{\mathrm{II}}^{O}$	$x_{2,4,5,6,8,9,10,11,12,13,14,15,18,20,21,23,24,26,28,29,30,31,32,33,34,36}$	$x_{4,6,9,14,18,26,29,33}$
Dq-MDTRS$_{\mathrm{I}}^{P}$	$x_{4,6,9,14,15,18,20,21,24,26,28,29,33,36}$	$x_{2,3,4,5,6,8,9,10,11,12,13,14,15,16,17,18,20,21,23,24,25,26,27,28,29,30,31,32,33,34,36}$
Dq-MDTRS$_{\mathrm{II}}^{P}$	$x_{4,5,6,9,14,15,18,20,21,24,26,28,29,33,36}$	$x_{2,4,5,6,8,9,10,11,13,14,15,18,20,21,23,24,26,28,29,31,33,34,36}$
Dq-MDTRS$_{\mathrm{I}}^{M}$	$x_{4,5,6,9,14,15,18,20,21,24,26,28,29,33,36}$	$x_{2,3,4,5,6,8,9,10,11,13,14,15,17,18,20,21,24,26,28,29,31,33,34,36}$
Dq-MDTRS$_{\mathrm{II}}^{M}$	$x_{2,4,5,6,8,9,10,11,14,15,18,20,21,24,26,28,29,31,33,34,36}$	$x_{2,4,5,6,8,9,10,11,14,18,20,21,24,26,28,29,31,33,34,36}$

Table 9
The Dq-MDT approximations of X with respect to $k = 2$, $\alpha = 0.7$ and $\beta = 0.5$.

Model	Lower approximation	Upper approximation
Dq-MDTRS$_{\mathrm{I}}^{O}$	$U - x_{1,7,19,22,35}$	$x_{4,5,6,9,14,18,20,24,26,29,33}$
Dq-MDTRS$_{\mathrm{II}}^{O}$	$x_{2,4,5,6,9,10,14,15,18,20,21,24,26,28,29,33,34,36}$	$x_{4,6,9,14,18,26,29,33}$
Dq-MDTRS$_{\mathrm{I}}^{P}$	$x_{4,6,9,14,15,18,20,21,24,26,28,29,33,36}$	$x_{2,4,5,6,8,9,10,11,13,14,15,18,20,21,23,24,26,28,29,31,33,34,36}$
Dq-MDTRS$_{\mathrm{II}}^{P}$	$x_{6,9,26}$	$x_{2,4,5,6,8,9,10,11,13,14,15,18,20,21,23,24,26,28,29,31,33,34,36}$
Dq-MDTRS$_{\mathrm{I}}^{M}$	$x_{4,5,6,9,14,15,18,20,21,24,26,28,29,33,36}$	$x_{2,4,5,6,8,9,10,11,14,15,18,20,21,24,26,28,29,31,33,34,36}$
Dq-MDTRS$_{\mathrm{II}}^{M}$	$x_{4,5,6,9,14,18,20,24,26,29,33}$	$x_{2,4,5,6,8,9,10,11,14,18,20,21,24,26,28,29,31,33,34,36}$

$neg_{\mathrm{I}}^{P}(X) = \{x_1, x_7, x_{19}, x_{22}, x_{35}\}$;
$Ubn_{\mathrm{I}}^{P}(X) = \{x_2, x_3, x_5, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{16}, x_{17}, x_{23}, x_{25}, x_{27}, x_{30}, x_{31}, x_{32}, x_{34}\}$;
$Lbn_{\mathrm{I}}^{P}(X) = \emptyset$.

For $\alpha = 0.5$, $\beta = 0.3$ and $k = 2$, these models have their own quantitative semantics for the relative and absolute degree quantification. Analogously, the thresholds can be determined by the real acquirements. Then, the decision rules can be simply achieved based on these studied decision mechanisms as follows:

(P_{I}^{P}) The patients $x_4, x_6, x_9, x_{14}, x_{15}, x_{18}, x_{20}, x_{21}, x_{24}, x_{26}, x_{28}, x_{29}, x_{33}$ and x_{36} are suffering from cold with respect to these diagnostic indexes and given parameters;
(N_{I}^{P}) The patients x_1, x_7, x_{19}, x_{22} and x_{35} are not suffering from cold regarding current diagnostic conditions;
(B_{I}^{P}) The patients $x_2, x_3, x_5, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{16}, x_{17}, x_{23}, x_{25}, x_{27}, x_{30}, x_{31}, x_{32}$ and x_{34} can not be diagnosed with respect to present information. We need to take a further diagnosis to make decisions.

Case 3. $\alpha + \beta > 1$. Consider the following loss function:

$$\lambda_{PP} = 0, \lambda_{PN} = 21,$$
$$\lambda_{BP} = 7, \quad \lambda_{BN} = 2,$$
$$\lambda_{NP} = 9, \quad \lambda_{NN} = 0.$$

According to the loss function, we can get that $\alpha = 0.7$, $\beta = 0.5$ that means $\alpha + \beta > 1$. We can get the optimistic, pessimistic and mean multigranulation decision-theoretic upper and lower approximations of concept X, respectively.

$\overline{\sum_{i=1}^{m} A_i}^{O}_{(0.7,\ 0.5)}(X) = \{x_4, x_5, x_6, x_9, x_{14}, x_{18}, x_{20}, x_{24}, x_{26}, x_{29}, x_{33}\}$,
$\underline{\sum_{i=1}^{m} A_i}^{O}_{(0.7,\ 0.5)}(X) = \{x_2, x_4, x_5, x_6, x_9, x_{10}, x_{14}, x_{15}, x_{18}, x_{20}, x_{21}, x_{24}, x_{26}, x_{28}, x_{29}, x_{33}, x_{34}, x_{36}\}$.
$\overline{\sum_{i=1}^{m} A_i}^{P}_{(0.7,\ 0.5)}(X) = \{x_2, x_4, x_5, x_6, x_8, x_9, x_{10}, x_{11}, x_{13}, x_{14}, x_{15}, x_{18}, x_{20}, x_{21}, x_{23}, x_{24}, x_{26}, x_{28}, x_{29}, x_{31}, x_{33}, x_{34}, x_{36}\}$,
$\underline{\sum_{i=1}^{m} A_i}^{P}_{(0.7,\ 0.5)}(X) = \{x_6, x_9, x_{26}\}$.
$\overline{\sum_{i=1}^{m} A_i}^{M}_{(0.7,\ 0.5)}(X) = \{x_2, x_4, x_5, x_6, x_8, x_9, x_{10}, x_{11}, x_{14}, x_{15}, x_{18}, x_{20}, x_{21}, x_{24}, x_{26}, x_{28}, x_{29}, x_{31}, x_{33}, x_{34}, x_{36}\}$,
$\underline{\sum_{i=1}^{m} A_i}^{M}_{(0.7,\ 0.5)}(X) = \{x_4, x_5, x_6, x_9, x_{14}, x_{18}, x_{20}, x_{24}, x_{26}, x_{29}, x_{33}\}$.

Analogously, we can get the lower and upper approximations of the designed double-quantitative multigranulation decision-theoretic rough set modes and results as shown in Table 9.

The lower and upper approximations of Dq-MDTRS$_{\mathrm{I}}^{O}$, Dq-MDTRS$_{\mathrm{II}}^{O}$, Dq-MDTRS$_{\mathrm{I}}^{P}$, Dq-MDTRS$_{\mathrm{II}}^{P}$, Dq-MDTRS$_{\mathrm{I}}^{M}$ and Dq-MDTRS$_{\mathrm{II}}^{M}$ with respect to $\alpha = 0.7$, $\beta = 0.5$ and $k = 2$ are revealed in Table 9. According to these achievements, we

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can get the rough regions of each rough set models. Without loss of generality, we take the Dq-MDTRS$_{\mathrm{I}}^{M}$ as an illustration, and the rough regions of this model are exhibited as follows:

$pos_{\mathrm{I}}^{M}(X) = \{x_4, x_5, x_6, x_9, x_{14}, x_{15}, x_{18}, x_{20}, x_{21}, x_{24}, x_{26}, x_{28}, x_{29}, x_{33}, x_{36}\}$;

$neg_{\mathrm{I}}^{M}(X) = \{x_1, x_3, x_7, x_{12}, x_{13}, x_{16}, x_{17}, x_{19}, x_{22}, x_{23}, x_{25}, x_{27}, x_{30}, x_{32}, x_{35}\}$;

$Ubn_{\mathrm{I}}^{M}(X) = \{x_2, x_8, x_{10}, x_{11}, x_{31}, x_{34}\}$;

$Lbn_{\mathrm{I}}^{M}(X) = \emptyset$.

For $\alpha = 0.7$, $\beta = 0.5$ and $k = 2$, these rough regions have their own quantitative semantics for the relative and absolute degree quantification. Based on these results, we can get that the rough regions are varied with the changes of thresholds. Furthermore, we can get the decision rules by using Dq-MDTRS$_{\mathrm{I}}^{M}$ model as follows:

(P_{I}^{M}) The patients $x_4, x_5, x_6, x_9, x_{14}, x_{15}, x_{18}, x_{20}, x_{21}, x_{24}, x_{26}, x_{28}, x_{29}, x_{33}$ and x_{36} are suffering from cold with respect to these diagnostic indexes and given parameters;

(N_{I}^{M}) The patients $x_1, x_3, x_7, x_{12}, x_{13}, x_{16}, x_{17}, x_{19}, x_{22}, x_{23}, x_{25}, x_{27}, x_{30}, x_{32}$ and x_{35} are not suffering from cold with respect to current diagnostic conditions;

(B_{I}^{M}) The patients $x_2, x_8, x_{10}, x_{11}, x_{31}$ and x_{34} can not be diagnosed with respect to present information. A further diagnosis is need to them.

With regard to one information system, the rough regions and decision rules rely on the parameters to solve different issues. Table 7, 8 and 9 exhibit the lower and upper approximations of proposed rough set models. According to these case studies, we can obtain that the rough regions and decision rules are varied with respect to different thresholds. For the same patient, the decision rule depends on the models and thresholds. The concept that we considered in these cases is cold patient set D_2. For the thresholds are $\alpha = 0.6$, $\beta = 0.4$ and $k = 2$, the decision rule indicate that the patients $x_4, x_5, x_6, x_9, x_{14}, x_{15}, x_{18}, x_{20}, x_{21}, x_{24}, x_{26}, x_{28}, x_{29}, x_{33}, x_{36}$ are suffering from cold. The patients x_4 and x_{36} are diagnosed with a cold in the model of Dq-MDTRS$_{\mathrm{I}}^{O}$, but they are not be treated as sick in original Table 2. We analyzed the symptoms of x_4 and x_{36}, the possibility of misdiagnosis is present in initial medicinal data. For the thresholds are $\alpha = 0.5$, $\beta = 0.3$ and $k = 2$, we can achieve that $x_4, x_6, x_9, x_{14}, x_{15}, x_{18}, x_{20}, x_{21}, x_{24}, x_{26}, x_{28}, x_{29}, x_{33}, x_{36}$ are diagnosed with a cold in the model of Dq-MDTRS$_{\mathrm{I}}^{P}$, we can get that the patient x_5 is health in this model but diseased in Dq-MDTRS$_{\mathrm{I}}^{O}$. Combining the symptoms of x_5 that shown in the Table 2, we can think that x_5 is not likely to catch a cold. For the thresholds are $\alpha = 0.7$, $\beta = 0.5$ and $k = 2$, the results show that the patients $x_4, x_5, x_6, x_9, x_{14}, x_{15}, x_{18}, x_{20}, x_{21}, x_{24}, x_{26}, x_{28}, x_{29}, x_{33}, x_{36}$ are suffering from cold in Dq-MDTRS$_{\mathrm{I}}^{M}$. These diagnostic results are consistent with previous diagnostic results in Dq-MDTRS$_{\mathrm{I}}^{P}$. The sets of health human are different in these models, but the people $x_1, x_7, x_{19}, x_{22}, x_{35}$ are health that are diagnosed by model Dq-MDTRS$_{\mathrm{I}}^{O}$, Dq-MDTRS$_{\mathrm{I}}^{P}$ and Dq-MDTRS$_{\mathrm{I}}^{M}$. The symptoms of them indicate that they are certainly healthy. Utilizing these models, we can perform some preliminary diagnostic analysis for patients. The diagnostic results for different models and thresholds are not completely consistent. Thus, we should choose an appropriate model and thresholds based on practical requirements in applications.

5. Conclusion

The graded rough set and decision-theoretic rough set serve as two generalized rough sets, which were utilized to measure the absolute and relative mutual information between the equivalence classes and stated concept in approximate space, respectively. In many circumstances, we often need to describe concurrently a target concept through multi granular structures and quantification criterions according to practical requirements of problem solving. In order to study double-quantification decision-theoretic approach which fuses the relative and absolute quantitative information in multigranulation approximate space, we introduce the idea of double-quantification decision-theoretic into the framework of multigranulation in this talk. Three pairs of double-quantitative multigranulation decision-theoretic rough set models are established by recombining the approximate operators of graded rough set and decision-theoretic rough set. They are Dq-MDTRS$_{\mathrm{I}}^{O}$, Dq-MDTRS$_{\mathrm{II}}^{O}$, Dq-MDTRS$_{\mathrm{I}}^{P}$, Dq-MDTRS$_{\mathrm{II}}^{P}$, Dq-MDTRS$_{\mathrm{I}}^{M}$ and Dq-MDTRS$_{\mathrm{II}}^{M}$, respectively. Several interesting properties of these models are addressed and the decision rules are also deduced based on the Bayesian decision method. These novel models perform a basic double quantification of the absolute information and relative information, and satisfy the quantitative completeness properties and exhibit strong fault tolerance capabilities. Based on the decision mechanisms, we designed a series of illustrations with respect to different combinations of parameters. This paper develops a basic framework of double-quantitative decision-theoretic rough set in multigranulation approximate space, and there are still some issues should be studied in our future work, for instance, double-quantitative multigranulation decision-theoretic rough set under dynamic granulation, double-quantitative multigranulation decision-theoretic rough set under different granulation with respect to different thresholds and the practical applications of double-quantitative multigranulation decision-theoretic rough set.

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